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Additive Number Theory for all Quadratic Functions.

By L. E. Dickson.

1. *Introduction.* We consider only those quadratic functions $q(x)$ which take integral values $\geqq 0$ for every integer $x \geqq 0$. The general such $q(x)$ is found in § 2.

One statement of our problem is to find the maximum error committed in affirming that every positive integer p is a sum of s such values of $q(x)$. Or, if $m(p)$ denotes the minimum of the differences between p and all such sums, what is the maximum of $m(p)$ for all p? The answer is evidently $\frac{1}{2}g$ or $\frac{1}{2}(g-1)$, according as g is even or odd, where g is the maximum gap in a table of all sums by s of values of $q(x)$ for integers $\geqq 0$.

Write $e_s = g - 1$. Then every p is r plus a sum of s values of $q(x)$, where r is chosen from $0, 1, \cdots, e_s$. For example, let $q(x)$ be $\frac{1}{2}(n-2)(x^2-x)+x$; then $e_4 = n-4$, as first proved by Cauchy. In words, every p is a sum of n polygonal numbers of n sides, all but four of which are chosen from 0 or 1. The case $n=4$ states that every p is a sum of four squares. However perfected, the method of Cauchy is usually applicable only when $s=4$.

For the case $s \geqq 5$ there is developed here a new method of remarkable power and simplicity, which leads readily to the determination of e_s for all $q(x)$. The two effective methods for $s=4$ (§§ 9, 10, and 21) are not only more laborious, but unfortunately require direct verification for very numerous initial values of p (unlike the method for $s>4$). While this would seem to require a table for each $q(x)$, the theory has been so developed that a few such tables suffice.

For each $q(x)$ we find the minimum L of $s+e_s$ for all $s \geqq 5$; this sum is the total number of summands sufficient for every p. We readily detect the cases in which $4+e_4 \geqq L$. It is only in the remaining cases that we discuss the difficult problem * to find e_4. Whenever $4+e_4$ is less than L it is exactly $L-1$. By examining also the cases $s=2$ and $s=3$, we find that the minimum of all $s+e_s$ is either L or $L-1$, except (§ 11) when $q(x) =$

* To be treated in Chicago theses.

$\frac{1}{2}m(x^2-x)+tx$, where m and t are positive integers such that $m \leqq t$. Then $e_s = t-1$ for all $s \geqq 3$.

An interesting case is that in which both 0 and 1 are values of $q(x)$. Then every p is a sum of values of $q(x)$. For this case the present problem was solved completely in the *Bulletin of the American Mathematical Society*, November-December, 1927; January-February, 1928; March-April, 1928. These are cited here as *Bulletin*, I, II, III.

Aid was provided by the Carnegie Institution for constructing tables and using them to verify the theorems for the cases below the limits obtained in the proofs.

2. THEOREM 1. *The following and no further quadratic functions $q(x)$ have integral values $\geqq 0$ for every* integer $x \geqq 0$: First,*

$$f(x) = \tfrac{1}{2}m(x^2-x)+tx+c, \tag{1}$$

where m, t, c are integers and $m > 0$, $t > 0$, $c \geqq 0$. Second, $f(x-k)$, where k is an integer $\geqq 1$, and $m \geqq t$.

Since $y = q(x)$ is a parabola whose vertex is its lowest point, there† is either a single minimum $q(k)$ of $q(0), q(1), \cdots$, or just two equal minima $q(k-1) = q(k)$. Hence there is a unique integer $k \geqq 0$ such that $q(x) > q(k)$ for every integer $x \geqq 0$ distinct from k and $k-1$; while $q(k-1) \geqq q(k)$ if $k \geqq 1$.

Write $f(x)$ for $q(x+k)$. Then $f(x)$ is an integer $\geqq 0$ for every integer $x \geqq -k$. Also, $f(x) > f(0)$ for every integer $x \geqq -k$ distinct from 0 and -1; while $f(-1) \geqq f(0)$ if $k \geqq 1$. It will be useful that $x = 0$ gives the minimum of $f(x)$ for all integers $x \geqq -k$.

Write $f(x) = ux^2+vx+c$. Take $x = 0, 1, 2$. Thus c, $u+v = t$, and $4u+2v$ are integers, and $u > 0$, $c \geqq 0$. Hence $2u$ is a positive integer m. Thus $f(x)$ is of the form (1). Since $f(1) = t+c > f(0) = c$, $t > 0$. Conversely, this $f(x)$ is evidently an integer $\geqq 0$ for every integer $x \geqq 0$.

In case $k \geqq 1$, $f(-1) = m-t+c \geqq f(0) = c$ implies $m \geqq t$, and con-

---

* It is only an apparent generalization to employ a function $g(x)$ which is an integer $\geqq 0$ for every integer $x \geqq r$, since $q(x) = g(x+r)$ is then an integer $\geqq 0$ for every integer $x \geqq 0$.

† Arithmetically, if $q'(\xi) = 0$, then $k = 0$ if $\xi \leqq 0$, $k = \xi$ if ξ is a positive integer, while if ξ lies between $p-1$ and p, where p is an integer $\geqq 1$, $k = p$ if $q(p-1)$ and $q(p)$ are equal, but, if they are distinct, $k = p-1$ or p according as $q(p-1)$ or $q(p)$ is the lesser.

versely. Then $f(-p) > f(0)$ for every $p > 1$. This is evident geometrically, and follows also from

$$f(x+1) = f(x) + mx + t, \quad f(-p) = f(-p+1) + mp - t. \tag{2}$$

3. *Symbols E_s, e_s; Reduction to $c = 0$.* The least value of $q(x) = f(x-k)$ for integers $x \geqq 0$ was seen to be $f(0) = c$. Hence the least sum of s values of $q(x)$ is sc. Let N be any integer $\geqq sc$. From N we subtract the maximum sum $\leqq N$ of s values of $q(x)$ for integers $x \geqq 0$ and denote the difference by $E_s(N)$. In case $E_s(N)$ has a finite maximum for all integers $N \geqq sc$, we shall denote that maximum by e_s. In other words, every integer $N \geqq sc$ is then a sum of e_s numbers 0 or 1 and s values of $q(x)$ for integers $x \geqq 0$. For example, since every integer $\geqq 0$ is a sum of four squares, $e_4 = 0$ for $q(x) = x^2 + 7$.

Write $h(x)$ for $q(x) - c$, and employ summations over s integers $x \geqq 0$. If $N = r + \Sigma q(x)$, where $N \geqq sc$, $0 \leqq r \leqq e_s$, then $A = N - sc = r + \Sigma h(x)$, $A \geqq 0$, and conversely. Hence $q(x)$ and $h(x)$ have the same e_s. Without loss of generality, we shall therefore take $c = 0$ henceforth. Then

$$q(x) = f(x-k) = \tfrac{1}{2}mx^2 + \tfrac{1}{2}nx + C, \tag{3}$$

$$n = 2t - m(1+2k), \quad C = \tfrac{1}{2}m(k^2+k) - tk. \tag{4}$$

The case $s > 4$ will be seen to depend on facts required in the harder case $s = 4$.

4. *Formulas for the Case $s = 4$.*

We investigate the existence of integers e_4, α, β, γ, δ, all $\geqq 0$, such that

$$A = q(\alpha) + q(\beta) + q(\gamma) + q(\delta) + r, \quad 0 \leqq r \leqq e_4, \tag{5}$$

for every integer $A \geqq 0$. Write

$$a = \Sigma\alpha^2, \quad b = \Sigma\alpha. \tag{6}$$

Then (5) is equivalent to

$$D = \tfrac{1}{2}ma + \tfrac{1}{2}nb + r, \quad D = A - 4C. \tag{7}$$

We shall use two lemmas * due to Cauchy.

LEMMA 1. *If a and b are positive odd integers such that $b^2 < 4a$ and*

$$b^2 + 2b + 4 > 3a, \tag{8}$$

then equations (6) *have solutions $\alpha, \beta, \gamma, \delta$ in integers $\geqq 0$.*

---

* Cf. Legendre, *Théorie des nombres*, ed. 3, vol. II, Nos. 624-30.

LEMMA 2. *If a and $b = 2B$ are positive even integers such that* (8) *holds and $a - B^2$ is a sum of three squares, then equations* (6) *have solutions $\alpha, \beta, \gamma, \delta$ in integers $\geqq 0$.*

In particular, let $a \equiv 2 \pmod 4$ and $b^2 < 4a$. Then $a - B^2$ is positive and is $\equiv 1$ or $2 \pmod 4$, whence it is a sum of three squares. Hence Lemma 1 holds also when a and b are positive even integers such that $a \equiv 2 \pmod 4$.

By (7), (8) is equivalent to

$$mb^2 + 2mb + 4m > 6D - 3nb - 6r,$$

which follows from the inequality obtained by suppressing $6r$. Multiplication by $4m$ yields

$$(9) \qquad (2mb + \tau)^2 > U, \quad U = \tau^2 + 4m(6D - 4m),$$

where $\tau = 2m + 3n$. This inequality holds if

$$(10) \qquad b > (U^{\frac{1}{2}} - \tau)/(2m), \quad U \geqq 0.$$

In $m^2(b^2 - 4a) < 0$, replace ma by its value from (7). This inequality follows from that with r replaced by e_4 and hence from

$$(11) \qquad (mb + 2n)^2 < 4V, \quad V = n^2 + 2mD - 2me_4.$$

This inequality holds if

$$(12) \qquad b < (2V^{\frac{1}{2}} - 2n)/m, \quad V \geqq 0,$$

and if $mb + 2n \geqq 0$. When $n \geqq 0$, the latter holds if $b > 0$. When $n < 0$, it holds by (10) if

$$(13) \qquad 4n + U^{\frac{1}{2}} - \tau \geqq 0,$$

and hence if

$$(14) \qquad 3D \geqq 2m - 2n - n^2/m \qquad (\text{if } n < 0).$$

Finally, $b > 0$ if $U^{\frac{1}{2}} \geqq \tau$. If $n < 0$, this follows from (13). But, if $n \geqq 0$, whence $\tau > 0$, it holds if and only if $U \geqq \tau^2$ and hence by (9) if

$$(15) \qquad D \geqq \tfrac{2}{3}m \qquad (\text{if } n \geqq 0).$$

There will be at least d positive integers between the limits (10) and (12) for b if

$$(16) \qquad 4V^{\frac{1}{2}} - U^{\frac{1}{2}} > P, \quad P = 2md - 2m + n.$$

The left member is $\geqq 0$ if

$$(17) \qquad 16V \geqq U.$$

Then (16) holds automatically if $P < 0$, and always if its square holds and hence if

$$F \equiv (2V + W)^2 - UV > 0, \quad 8W \equiv U - P^2. \tag{18}$$

By the minor conditions we shall mean $U \geqq 0$, (17) and either (14) or (15). We have

$$\begin{aligned} &16V - U = 8mD - 32me_4 + \phi, \\ &\phi = 12m^2 - 12mn + 7n^2 = 4n^2 + 3(2m - n)^2. \end{aligned} \tag{19}$$

By (9) and (19), $U \geqq 0$ and (17) holds if

$$D \geqq \tfrac{2}{3}m, \quad D \geqq 4e_4. \tag{20}$$

The *reduced* minor conditions are (20) if $n \geqq 0$, but (14) and (20) if $n < 0$.

PART I. THE CASE $k = 0$, §§ 5-12.

We investigate sums of s values of $f(x)$. The case $s \geqq 5$ is treated in §§ 5-7, the case $s = 4$ in §§ 8-10, the case $s = 3$ in § 11, and the case $s = 2$ in § 12.

5. THEOREM 2. *If* $k = 0$, $m \geqq (s-2)t$, $s \geqq 5$, *then* $e_s = m - (s-3)t - 1$.

No smaller value of e_s will serve. For, if $B = 2m + 4t - 1$, the maximum sum $\leqq B$ of s values

$$0, \quad t, \quad m + 2t, \quad 3m + 3t, \quad 6m + 4t, \quad 10m + 5t, \quad \cdots \tag{21}$$

of $f(x)$ is $M = m + (s+1)t$. In fact, no summand exceeds $m + 2t$. If one summand is $m + 2t$, the others are 0 or t, and the largest such sum is $m + 2t + (s-1)t = M$. The largest sum of s values 0 and t is $st < M$. Hence $E_s(B) = B - M$ has the value stated for e_s.

We are to prove that every integer $N \geqq 0$ is a sum of s values of $f(x)$ and e_s numbers 0 or 1. Write $N = A + R$, where

$$A = \sum_4 f(x), \quad R = r + \sum_{s-4} f(x), \quad 0 \leqq r \leqq e_s. \tag{22}$$

We can find a value $\leqq 2m - 1$ of R which is congruent to any assigned integer modulo m. We add $0, 1, \cdots, t-1 \leqq e_s$ to each of $0, t, \cdots, (s-4)t$. But we add $0, 1, \cdots, e_s$ to $m + 2t + (s-5)t$. We obtain m integers R which are congruent modulo m to the consecutive integers $0, \cdots, m-1$. The largest such R is $2m - 1$.

We shall treat the first equation (22) by the method * of § 4 with $e_4 = 0$. By (4) with $k = 0$, $n = 2t - m$, $C = 0$. Then $D = A$ and (7) becomes

$$A = \tfrac{1}{2}m(a-b) + tb. \tag{23}$$

Let the difference between the limits (10) and (12) on b exceed $d = 2$. Then there is an odd value of b between these limits. We proved that we can choose $R \leqq 2m - 1$ so that $A = N - R$ is congruent modulo m to any assigned integer. As the latter we take tb. Then (23) determines an integer $\frac{1}{2}(a-b)$ and hence an odd integer a, which is positive since $4a > b^2$. By (9)-(18).

$$P = 2m + n, \quad W = 3mD - 2m^2 + mn + n^2,$$
$$F = m^2(D - 2m - 5n - 4e_4)^2 - 6m\,(n + e_4)(2m + n)^2, \tag{24}$$
(if $d = 2$).

Since $m \geqq 3t$, n is negative. Hence $F > 0$ for every $D = A$. Here (14) becomes $3A \geqq 3m - 4t^2/m$. This and (20) hold if $A \geqq m$.

Hence if $N \geqq 2m - 1 + m$, N is a sum of e_s numbers 0 or 1 and s values of $f(x)$. We next verify this also when $N < 3m - 1$. We employ the sums by s of 0 and t; $m + 2t$ plus their sums by $s - 1$; and $2(m + 2t)$ plus their sums by $s - 2$. Hence sums by s of values of $f(x)$ include.

$$0, t, \cdots, st; \quad m + 2t, \cdots, m + (s+1)t;$$
$$2m + 4t, \cdots, 2m + (s+2)t.$$

To the last number in each of these three sets we add $0, 1, \cdots, e_s$; to all others we add only $0, 1, \cdots, t - 1$. We obtain the consecutive integers from 0 to $3m + 5t - 1$.

6. THEOREM 3. *If* $k = 0$, $m < (s-2)t$, $s \geqq 5$, *then* $e_s = t - 1$.

No smaller e_s will serve since 0 is the only value $< t$ of $f(x)$. To $0, t, \cdots, (s-4)t$, $m + 2t + (s-5)t$, we add $0, 1, \cdots, t - 1$ and obtain $(s-2)t > m$ integers R which are congruent modulo m to the consecutive integers from 0 to $(s-2)t - 1$. The largest such R is $\rho = m + (s-2)t - 1$.

First, let $m \geqq 2t$. Then $n \leqq 0$. As in § 5, every integer $N \geqq \rho + m$, is a sum of $t - 1$ numbers 0 or 1 and s values of $f(x)$, and the same is true if $N < \rho + m$.

---

* After the paper was in type, I found that it is better to take as e_4 the maximum R, add R at the end of (23), and choose R so that $tb + R \equiv A \pmod m$. A like remark applies to all theorems with $s > 4$ (§§ 6, 13, 14, 29, 30, 33, 34).

Second, let $t \leqq m < 2t$. Values of R include $0, 1, \cdots, t-1$ and the sum of t and $0, 1, \cdots, m-t \leqq t-1$. These $m+1$ values $\leqq m$ of R include a complete set of residues modulo m. We replace n by its value $2t-m$ in (24) and note that $D = A$. We get

$$F = m^2w - 48mt^3 - 24m^2t^2 + 12m^3t + 6m^4, \quad w = A + 3m - 10t.$$

Let $A \geqq 7m + 10t$. Then $w \geqq 10m$, $m^2w^2 > 96m^4 \geqq 96mt^3$,

$$F \geqq 24mt^2(2t - m) > 0.$$

Hence every integer $N \geqq 8m + 10t$ is a sum of e_s numbers 0 or 1 and s values of $f(x)$. To verify this also for $N \leqq 9m + 11t$, note that the sums by four of the values (21) of $f(x)$ include

$$\begin{array}{ll} & 0,\ t,\ 2t,\ 3t,\ 4t;\quad m+3t,\ 4t,\ 5t;\quad 2m+4t,\ 5t,\ 6t; \\ (25) & 3m+5t,\ 6t,\ 7t;\quad 4m+6t,\ 7t,\ 8t;\quad 5m+7t,\ 8t;\quad 6m+7t,\ 8t,\ 9t; \\ & 7m+8t,\ 9t;\quad 8m+8t,\ 9t,\ 10t;\quad 9m+9t,\ 10t, \end{array}$$

whose successive gaps are t or $m - t < t$.

Finally, it remains only to prove the following theorem: *If $m < t$, every integer $N \geqq 0$ is a sum of five values of $f(x)$ and $t-1$ numbers 0 or 1.* We seek five integers $x \geqq 0$ such that

$$N = A + R, \quad A = \sum_4 f(x), \quad R = f(x) + r, \quad 0 \leqq r \leqq t-1.$$

To the value 0 of $f(x)$ we add all values of r. To the value t of $f(x)$ we add only the values $0, 1, \cdots, 2m-t-1 < t-1$ of r. We obtain the consecutive integral values $0, 1, \cdots, 2m-1$ of R. Hence we can choose $R < 2m$ so that A is congruent modulo $2m$ to any assigned integer.

If b is odd, we take $A \equiv tb \pmod{2m}$. Then (23) yields an integral value of $\frac{1}{2}(a-b)$ and hence an odd integral value of a. But if b is even, we take

$$b = 2B, \quad A \equiv 2tB + m(1-B) \pmod{2m}.$$

Then $A - 2tB + mB$ is the product of m by an odd integer ϵ. Write $a = 2\epsilon$. Then (23) holds. Since $a \equiv 2 \pmod 4$, Lemma 2 is applicable.

Since we may use either an even or odd value of b, we may employ § 4 with $d = 1$, $e_4 = 0$, $C = 0$, $D = A$. Then $P = n$ is positive. We have

$$2W = 6mA - 3m^2 + 3mn + 2n^2,$$

and find that

$$4F = m^2(2A + 3m - 3n - 8e_4)^2 + 12mn^2(m - n - 2e_4).$$

Replacing n by its value $2t-m$, we get

$$F=m^2w^2+6(-4mt^3+8m^2t^2-5m^3t+m^4), \quad w=A+3m-3t.$$

First, let $2m\geqq t$. If $A\geqq 5m+9t$, then $w\geqq 8m+6t$,

$$mw\geqq 8m^2+6mt\geqq 2t^2+3t^2, \quad m^2w^2\geqq 25t^4>24mt^3,$$

since $t>m$. Then $F>0$. Hence the theorem is proved for integers $N\geqq 7m+9t$. To verify it for $N\leqq 9m+11t$, we omit from (25) the initial values $m+3t$, $2m+4t$, $\cdots$, $9m+9t$ from each block and see that all gaps are now t or $m<t$.

There remains the case $2m<t$. Then $3m^2t^2>6m^3t$. Hence $F>0$ if $m^2w^2\geqq 24mt^2-30m^2t^2$. Let $5m\geqq t$. Then $120m^2t^2\geqq 24mt^3$. Thus $F>0$ if $m^2w^2\geqq(120-30)m^2t^2$ and hence if $w\geqq 10t$, i. e., if $A\geqq -3m+13t$. This holds if $A\geqq -3m+13t+2(5n-t)=7m+11t$, and hence if $N\geqq 9m+11t$. For the smaller N's, the theorem was verified above.

There remains the case $5m<t$. Then $m^2t^2>5m^3t$, so that $F>0$ if $m^2w^2\geqq 24mt^3-42m^2t^2$. To dispose of the subcase $pm\geqq t$, we proceed as follows. Then $24pm^2t^2\geqq 24mt^3$. Thus $F>0$ if $m^2w^2\geqq q^2m^2t^2$, where $q^2\geqq 24p-42$, and hence if $A\geqq -3m+(q+3)t$, or

$$N\geqq -m+(q+3)t. \tag{26}$$

Since $2(pm-t)\geqq 0$, (26) holds if

$$N\geqq(2p-1)m+(q+1)t.$$

Take $p=18$. The least q is 20. It remains to investigate the integers $N<35m+21t$. Those $\leqq 9m+11t$ were discussed above. It suffices to employ the following sums by four of numbers (21):

$$9m+10t;\ 14m+10t,\ 11t,\ 12t;\ 19m+12t,\ 13t,\ 14t;\ 24m+14t,\ 15t,\ 16t;$$
$$29m+16t,\ 17t;\ 34m+17t,\ 18t;\ 38m+18t,\ 19t;\ 43m+19t,\ 20t,$$

whose gaps are t, $4m$ and $5m<t$.

There now remains the case $18m<t$. We now proceed by induction on l from the case $2lm<t$ to the case $2(l+1)m<t$. Hence let $2lm<t$, $pm\geqq t$, where $p=2(l+1)$, and l is an integer $\geqq 9$. We may take $q=2l+3$ since $q^2\geqq 24p-42$ for $l\geqq 9$. Then

$$f(2l)=l(2l-1)m+2lt\geqq(2l+6)t,$$

since

$$12(l+1)m\geqq 6t, \quad l(2l-1)>12(l+1),$$

for $l\geqq 9$. Hence $f(2l)$ exceeds the limit (26), whence the theorem holds for all integers $\geqq f(2l)$.

To verify it for integers $< f(2l)$, we apply the following result for $\lambda = 2l$, $2lm < t$.

LEMMA 3. *In a table of all sums by two of values of $f(x)$, all gaps are $\leqq t$ to $f(\lambda)$ if $(\lambda - 1)m \leqq t$.*

For a given g, the sums $gt + s_1 m$, $gt + s_2 m$, $\cdots$, arranged in ascending order of magnitude, will be said to form the *block of rank g*. That of rank 4 is composed of $4t + 2m$, $4t + 3m$, $4t + 6m$. The block of rank 2ξ is composed of

$$(27) \qquad f(\xi + \delta) + f(\xi - \delta) = 2\xi t + (\delta^2 + \xi^2 - \xi)m \quad (\delta = 0, 1, \cdots, \xi).$$

When δ is changed to $\delta + 1$, the increment to this sum is $(2\delta + 1)m$. Hence the largest number of the block is evidently $f(2\xi)$, whence $2\xi \leqq \lambda$. Hence the maximum gap within this block is $(2\xi - 1)m \leqq t$. The numbers of the block of rank $2\xi - 1$ are

$$(28) \quad f(\xi + \delta - 1) + f(\xi - \delta) = (2\xi - 1)t + (\delta^2 - \delta + \xi^2 - 2\xi + 1)m \quad (\delta = 1, \cdots, \xi).$$

When δ is changed to $\delta + 1$, the increment to this sum is $2\delta m$. Hence the largest number of this block is given by $\delta = \xi$ and is $f(2\xi - 1)$. Thus $2\xi - 1 \leqq \lambda$, and the maximum gap within this block is $2(\xi - 1)m \leqq t$.

Subtracting the largest number $f(2\xi - 1)$ in the block of rank $2\xi - 1$ from the least number $2f(\xi)$ in the block of rank 2ξ, we get $t - m(\xi - 1)^2$, which is either $\leqq 0$ or is positive and $\leqq t$. Hence either these blocks overlap or the gap between them is $\leqq t$. Likewise, if we subtract the largest number $f(2\xi - 2)$ of the block of rank $2\xi - 2$ from the least number of that of rank $2\xi - 1$, we get $t - (\xi - 1)(\xi - 2)m \leqq t$ for $\xi \geqq 2$.

7. THEOREM 4. *Let $k = 0$ and write L for the least $s + e_s$ for $s \geqq 5$. When $t = 1$, $L = m + 2$ if $m \geqq 3$, $L = 5$ if $m = 1$ or 2. When $t > 1$, write $m = ut + v$, $0 \leqq v < t$; then $L = u + t + 1$ if $v = 0$, $u \geqq 3$, $L = u + t + 2$ if $v \geqq 1$, $u \geqq 3$, $L = t + 4$ if $u = 0$, 1, or 2.*

First, let $m \geqq (s - 2)t$, $s \geqq 5$. By § 5,

$$s + e_s = m - s(t - 1) + 3t - 1.$$

If $t = 1$, this is $m + 2$ and is independent of s. But if $t > 1$, the sum is a minimum for the maximum permissible value $u + 2$ of s, since $u \geqq s - 2$. The minimum is $\lambda = v + u + t + 1$ if $u \geqq 3$.

Second, let $m < (s-2)t$, $s \geqq 5$. By § 6, $s + e_s = s + t - 1$ is least for the least permissible s. If $t = 1$, the least s is 5 if $m < 3$, but is $m + 3$ if $m \geqq 3$. Let $t > 1$. Then $u \leqq s - 3$. When $u = 0$, 1, or 2, the least permissible s is 5. When $u \geqq 3$, the least permissible s is $u + 3$, and the least $s + e_s$ is $u + t + 2$. This is $\geqq \lambda$ if $1 \leqq v$.

CASES IN WHICH $k = 0$, $4 + e_4 < L$, §§ 8-10.

8. When $t = 1$, it is known (*Bulletin*, I) that $4 + e_4$ is $m + 2$ if $m \geqq 2$, but is 4 if $m = 1$. Henceforth let $t > 1$.

THEOREM 5. *If $k = 0$, $t > 1$, $m > 2t$, then $4 + e_4 \geqq L$. But if $m = 2t$, then $4 + e_4 = t + 3 = L - 1$.*

No sum of four values (21) of $f(x)$ lies between $m + 5t$ and $2m + 4t$, since the value $3m + 3t$ exceeds the latter. Write $m = ut + v$. Then $u \geqq 2$,

$$4 + E_4(2m + 4t - 1) = 4 + m - t - 1 = ut + v - t + 3 \equiv \mu.$$

First, let $v = 0$, $u \geqq 3$. Then $\mu - L = (u-2)(t-1) > 0$.

Second, let $v \geqq 1$, $u \geqq 3$. Then $\mu - L = (u-2)(t-1) + v - 1 > 0$.

Third, let $u = 2$. Then $\mu - L = v - 1$. There remains only the case $u = 2$, $v = 0$, whence $m = 2t$. Then $f(x) = tx^2$, whence every multiple of t is a sum of four values of $f(x)$. Evidently $e_4 = t - 1$.

9. THEOREM 6. *If $k = 0$, $m \leqq t$, then $e_4 = t - 1$.*

If $m = t$, $f(x)$ is the product of the general triangular number by t. Hence every multiple of t is a sum of four (in fact, three) values of $f(x)$. Hence $e_4 = t - 1$.

For $k = 0$, (4), (7), (9), and (11) give

$$n = 2t - m, \quad C = 0, \quad D = A = m(a-b)/2 + tb + r, \tag{29}$$

$$U = 24mA + 36t^2 - 12tm - 15m^2, \tag{30}$$

$$V = 2mA + 4t^2 - 6tm + m^2 + 2m. \tag{31}$$

For convenience write $d = 2\delta$. By (16) and (18), we get

$$\begin{aligned} F &= m^2\omega^2 - Jt^3m - Kt^2m^2 - Ltm^3 - Mm^4 \\ &\quad + 24t^2m + (96\delta - 72)tm^2 + (96\delta^2 - 144\delta + 54)m^3, \\ \omega &= A - 14\delta t - (14\delta^2 - 21\delta + 4)m + 4, \quad J = 48\delta + 24, \\ K &= 240\delta^2 - 120\delta - 72, \quad L = 384\delta^3 - 624\delta^2 + 180\delta + 54, \\ M &= 192\delta^4 - 576\delta^3 + 540\delta^2 - 162\delta. \end{aligned} \tag{32}$$

(I) Let $2m \geqq t > m$. Since the values $0, 1, \cdots, t-1$ include a complete set of residues modulo m, the same is true of the values of $tb + r$ for any fixed b. There will exist an odd b between its limits if $d = 2$. We take $tb + r \equiv A \pmod{m}$. Then (29) determines an integer $(a-b)/2$ and hence an odd integer a.

Since $\delta \geqq 1$, the last three terms of F in (32) are positive. Here $\delta = 1$. Then $L = M = -6$. Hence $F > 0$ if

$$m^2\omega^2 \geqq 72t^3m + 48t^2m^2, \quad \omega = A - 14t + 3m + 4.$$

Multiply $2m \geqq t$ by $72t^2m$. Thus $F > 0$ if $m^2\omega^2 \geqq (144 + 48)t^2m^2$ and hence if $\omega \geqq 14t$, and therefore if

$$A \geqq 13(2m - t) + 28t - 3m = 23m + 15t.$$

But the gaps are $\leqq t$ to $23m + 15t$ in a table T_0 of sums by four of values of $f(x)$. Such a table is derived from Table I in *Bulletin*, I, by multiplying all terms free of m by t.

(II) Let $2m < t$. We may employ a single odd b as in (I) or a single even b. For, if $b \equiv 2 \pmod 4$, $tb + r \equiv A \pmod{2m}$ holds for one of the available values $0, 1, \cdots, t-1$ of r. Then (29) determines an even integer $(a-b)/2$, whence $a \equiv 2 \pmod 4$, and Lemma 2 is applicable. But, if $b \equiv 0 \pmod 4$, $tb + r \equiv A + m \pmod{2m}$ holds for one of the available values of r. Then (29) determines an odd integer $(a-b)/2$, whence $a \equiv 2 \pmod 4$.

Hence $d = 1$, $\delta = \frac{1}{2}$. By (32),

$$F = m^2\omega^2 - 48t^3m + 72t^2m^2 - 36tm^3 + 6m^4 + 6m^3 + 24tm(t - m),$$

where $\omega = A - 7t + 3m + 4$. Since $18t^2m^2 - 36tm^3 = 18tm^2(t - 2m) > 0$, $F > 0$ if $m^2\omega^2 \geqq 48t^3m - 54t^2m^2$.

Consider the case $5m \geqq t$. Then $240t^2m^2 \geqq 48t^3m$ and the preceding condition holds if $m^2\omega^2 \geqq ht^2m^2$, $h = 240 - 54 = 186 < 14^2$ and hence if $\omega \geqq 14t$. Thus $F > 0$ if

$$A \geqq 6(5m - t) + 21t - 3m - 4 = 27m + 15t - 4.$$

For $A \leqq 27m + 16t$, $E_4(A) \leqq t - 1$ by table T_0.

(III) There remains the case $5m < t$. Then $8t^2m^2 > 40tm^3$, and $F > 0$ if $m^2\omega^2 \geqq 48t^3m - 64t^2m^2$. Consider the subcase $pm \geqq t$. Then $48pt^2m^2 \geqq 48t^3m$ and $F > 0$ if $\omega \geqq qt$, where $q^2 \geqq 48p - 64$, and hence if

$$A \geqq a, \quad a = (q + 7)t - 3m - 4. \tag{33}$$

Take $p = 34$. The least q is 40. Then (33) holds if

$$A \geqq 6(34m - t) + a = 201m + 41t - 4.$$

For smaller A's, $E_4(A) \leqq t - 1$ by table T_0.

(IV) For the final case $34m < t$, we proceed by induction on l from the case $2lm < t$ to the case $pm < t$, $p = 2(l+1)$. Hence let $2lm < t$, $pm \geqq t$, $l \geqq 17$. We may take $q = 2l + 7$ since $q^2 \geqq 96l + 32$. Then $f(2l) \geqq (2l + 14)t > a$. In fact,

$$l(2l - 1)m > 14(2l + 2)m \geqq 14t,$$

since $(4l - 29)^2 \geqq 39^2 > 29^2 + 8 \cdot 28$. This proves the theorem for numbers $\geqq f(2l)$. It follows from Lemma 3 for numbers $\leqq f(2l)$.

10. THEOREM 7. *If* $k = 0$, $t < m$, $7t \geqq 4m$, *then* $e_4 = t - 1$. *If* $7t < 4m < 8t$, *then* $e_4(A) = t - 1$ *for* $A \leqq 400m + 58t$.

First, let $t < m$, $2m \leqq 3t$. We assign to r the values $0, 1, \cdots, t-1$, and to b two odd values β, $\beta + 2$. Then the residues modulo m of the values of $tb + r$ include $t\beta + j$ ($j = 0, 1, \cdots, m - 1$) and hence a complete set of residues. In fact, $t(\beta + 2) \equiv t\beta + h \pmod{m}$, where $h = 2t - m$ is positive and $< t$. The values of r and $h + r$ together include $0, 1, \cdots, 3t - m - 1 \geqq m - 1$. The desired two odd values of b will occur between the limits for b if their difference exceeds $d = 4$. Since $\delta = 2$, $F > 0$ by (32) if

$$m^2\omega^2 \geqq 120t^3m + 648t^2m^2 + 990tm^3 + 300m^4,$$

where $\omega = A - 28t - 18m + 4$. Multiply $m > t$ by $120t^2m$, $3t \geqq 2m$ by $495\,tm^2$, and $9t^2 \geqq 4m^2$ by $75m^2$. Hence $F > 0$ if

$$m^2\omega^2 \geqq gt^2m^2, \quad g = 120 + 648 + 1485 + 675 = 2928 < 55^2,$$

and hence if $\omega \geqq 55t$. This holds if

$$A \geqq 57(m - t) + 83t + 18m = 75m + 26t.$$

Here $n > 0$ and the reduced minor conditions (20) hold if $A \geqq 4t$. Using only $m < 2t$, we see that all gaps are $\leqq t$ down to $75m + 25t$ in table T_0.

Second, let $2m > 3t$, $7t \geqq 4m$. Give to b the odd values β, $\beta + 2$, $\beta + 4$, $\beta + 6$. Then the values of $tb + r$ are congruent modulo m to the sum of $t\beta$ and the values of r, $h + r$, $2h + r$, $3h + r$, which include the consecutive

numbers from 0 to $3h+t-1 \geqq m-1$. Thus $d=8$, $\delta=4$. By (32), $F>0$ if

$$m^2\omega^2 \geqq 216t^3m + 3288t^2m^2 + 15366tm^3 + 20280m^4,$$

where $\omega = A - 56t - 144m + 4$. Multiply $2m > 3t$ by $72t^2m$, $7t \geqq 4m$ by $3842tm^2$, and $49t^2 \geqq 16m^2$ by $1268m^2$. Hence $F>0$ if $\omega \geqq 305t$, since

$$305^2 > 92458 = 144 + 3288 + 26894 + 62132.$$

Thus $F>0$ if

$$A \geqq 361t + 144m + 103(2m-3t) = 350m + 52t.$$

When $m<2t$, we saw that all gaps are $\leqq t$ in T_0 to $75m+25t$; this is true also to $115m+27t$. From thence to $354m+55t$, we employ only $m<3t$ and find that all gaps are $\leqq t$ in T_0 and the following extension of it.

TABLE T_0.

All Sums of Four Polygonal Numbers.

$200m$+25—27, 29, 30, 32—35, 37—42; $201m$+27, 28, 31—37, 39—42; $202m$+28—42; $203m$+28—33, 35—41; $204m$+30, 32—42; $205m$+26—28, 31—37, 39—42; $206m$+28—31, 33—35, 37—42; $207m$+28—33, 35—41; $208m$+29—42; $209m$+31—37, 39—42; $210m$+21—24, 30—39, 41—43; $211m$+23—25, 27—33, 35—41, 43; $212m$+25, 26, 29, 30, 32—43; $213m$+24—27, 31—37, 39—43; $214m$+26, 27, 30—43; $215m$+28, 31—33, 35—41, 43; $216m$+25—31, 33—43; $217m$+27—29, 31—37, 39—43; $218m$+28—30, 33—43; $219m$+28—33, 35—41, 43; $220m$+26—28, 30, 32—44; $221m$+28, 29, 31—37, 39—44; $222m$+29—31, 33—35, 37—44; $223m$+29—31, 35—41, 43, 44; $224m$+31—44; $225m$+27—29, 31—37, 39—44; $226m$+29—39, 41—44; $227m$+31—33, 35—41, 43, 44; $228m$+30, 31, 33—44; $229m$+32—37, 39—44; $230m$+31, 32, 34—44; $231m$+22—25, 28—33, 35—41, 43—45; $232m$+24—26, 30, 31, 33—45; $233m$+26, 27, 32, 34—37, 39—45; $234m$+25—28, 31, 32, 34—43, 45; $235m$+27, 28, 30—33, 36—41, 43—45; $236m$+29, 32—39, 41—45; $237m$+26—29, 31—37, 39—45; $238m$+28—31, 33—35, 37—45; $239m$+30—32, 35—41; 43—45; $240m$+29—45; $241m$+27—29, 31—37, 39—45; $242m$+29, 30, 34—39, 41—46; $243m$+30—32, 35—41, 43—46; $244m$+30—46; $245m$+31—37, 39—45; $246m$+28—46; $247m$+30—33, 35—41, 43—46; $248m$+32—35, 37—46; $249m$+31—34, 36, 37, 39—45; $250m$+33—43, 45, 46; $251m$+32, 33, 35—41, 43—46; $252m$+29—46; $253m$+23—26, 31, 32, 34—37, 39—45, 47; $254m$+25—27, 33, 35, 37—47; $255m$+27, 28, 31—33, 35—41, 43—47; $256m$+26—29, 32—47; $257m$+28, 29, 34—37, 39—45, 47; $258m$+30, 33—39, 41—47; $259m$+27—32, 35—41, 43—47; $260m$+29—33, 37—47; $261m$+31, 34—37, 39—45, 47; $262m$+30—47; $263m$+28—30, 32, 35, 36, 38—41, 43—47; $264m$+30, 31, 37—48; $265m$+31—37, 39—45, 47, 48; $266m$+31—43, 45—48; $267m$+31—33, 35—41, 43—48; $268m$+29—31,

33—48; $269m+31$—37, 39—45, 47, 48; $270m+33$—35, 37—48; $271m+32$, 33, 35—41, 43—48; $272m+34$—36, 38—48; $273m+33$—37, 39—45, 47, 48; $274m+30$—39, 41—48; $275m+32$, 33, 35—41, 43—48; $276m+24$—27, 32—43, 45—49; $277m+26$—28, 33—37, 39—45, 47—49; $278m+28$, 29, 34—47, 49; $279m+27$—30, 35—41, 43—49; $280m+29$, 30, 34—49; $281m+31$—37, 39—45, 47—49; $282m+28$—31, 33, 34, 36—43, 45—49; $283m+30$—32, 35—41, 43—49; $284m+32$, 34—38, 40—49; $285m+31$—33, 36, 37, 39—45, 47—49; $286m+29$—31, 33—35, 37—49; $287m+31$, 32, 35—41, 43—49; $288m+32$—50; $289m+32$—37, 39—45, 47—50; $290m+34$, 35, 37—39, 41—50; $291m+30$—32, 35—41, 43—49; $292m+32$—50; $293m+34$, 35, 37, 39—45, 47—50; $294m+33$, 34, 36, 38—47, 49, 50; $295m+35$—41, 43—49; $296m+34$, 35, 37—50; $297m+31$—36, 39—45, 47—50; $298m+33$—43, 45—50; $299m+35$—38, 40, 41, 43—49; $300m+25$—28, 34, 35, 37—39, 41—51; $301m+27$—29, 35—37, 39—45, 47—51; $302m+29$, 30, 37—51; $303m+28$—31, 35—41, 43—49, 51; $304m+30$—34, 37—51; $305m+32$, 34, 35, 39—45, 47—51; $306m+29$—32, 36—39, 41—51; $307m+31$—33, 35—49, 51; $308m+33$—51; $309m+32$—37, 39—45, 47—51; $310m+30$—32, 34, 36—47, 49—51; $311m+32$, 33, 37—41, 43—49, 51; $312m+33$—35, 37—52; $313m+33$—36, 39—45, 47—52; $314m+35$, 37—43, 45—52; $315m+31$—33, 36—41, 43—49, 51, 52; $316m+33$—36, 38—52; $317m+35$, 36, 40—45, 47—52; $318m+34$, 35, 37—52; $319m+35$—41, 43—49, 51, 52; $320m+35$—38, 40—47, 49—52; $321m+32$—37, 39—45, 47—52; $322m+34$—39, 41—52; $323m+36$, 38—41, 43—49, 51, 52; $324m+35$—52; $325m+26$—29, 36, 37, 39—45, 47—53; $326m+28$—30, 37—39, 41—47, 49—51, 53; $327m+30$, 31, 36—41, 43—49, 51—53; $328m+29$—35, 38—53; $329m+31$, 32, 35, 36, 40—45, 47—53; $330m+33$, 36—53; $331m+30$—33, 35—40, 43—53; $332m+32$—34, 37—53; $333m+34$, 39—53 $334m+33$—35, 37—53; $335m+31$—33, 35, 39—41, 43—49, 51—53; $336m+33$—53; $337m+34$—37, 39—45, 47—54; $338m+34$—36, 38, 39, 41, 42, 44—54; $339m+36$—38, 40, 41, 43—54; $340m+32$—34, 37, 39, 41—43, 45—54; $341m+34$—37, 40—45, 47—53; $342m+36$—54; $343m+35$—41, 43—49, 51—54; $344m+37$—54, $345m+35$—37, 39—54; $346m+33$—43, 45—54; $347m+35$, 36, 38—49, 51—54; $348m+37$—54; $349m+36$, 37, 39—45, 47—54; $350m+37$, 38, 41—55; $351m+27$—30, 38—41, 43—55; $352m+29$—31, 37—55; $353m+31$, 32, 34—36, 39, 40, 42—45, 47—55; $354m+30$—33, 36—39, 41—55.

Third, let $7t < 4m < 8t$. Is $e_4 = t-1$? Let l be the least integer for which $t(2l-1) \geqq ml$. By using the odd values $\beta, \beta+2, \cdots, \beta+2l-2$ of b, we get $d = 2l$, and the method of § 21 gives the same d. The resulting limit for A is beyond our table. An incomplete extension of the latter to $400m + 58t$ was made quickly by adding 0, 1, and * occasionally $m+2$ to the tabulated sums by three of values of $f(x)$. When $m < 3t$, the gaps are $\leqq t$ in this extension.

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* Also $3m+3$ to $383m+40$ and $397m+49$, 50; and $6m+4$ to $386m+44$, $394m+50$.

All the results in §§ 8-10 may be combined into

THEOREM 8. *Let $k=0$ and L be the least $s+e_s$ for $s \geqq 5$. If $m>2t$, then $4+e_4 \geqq L$. But $4+e_4=t+3=L-1$ if $m=2t$ or $4m \leqq 7t$ and doubtless also if $7t<4m<8t$.*

11. THEOREM 9. *If $k=0$, $m>t$, $3+e_3$ is not less than the minimum $s+e_s$ for $s \geqq 4$. But if $m=t$, $e_3=t-1$, while if $m<t$, $e_3(A) \leqq t-1$ for $A \leqq 400m+58t$.*

The sums by three of the values of $f(x)$ are

$$0,\ t,\ 2t,\ 3t;\ m+2t,\ m+3t,\ m+4t;\ 2m+4t,\ 2m+5t;$$
$$3m+3t, \cdots, 3m+6t;\ 4m+5t,\ 4m+6t;\ 5m+7t;$$
$$6m+4t, \cdots, 6m+7t;\ 7m+6t, \cdots;\ 21m+7t, \cdots, 21m+12t;$$
$$22m+9t,\ 22m+10t,\ 22m+12t,\ 22m+13t;$$
$$23m+11t,\ 23m+13t;\ 24m+10t, \cdots.$$

If $m \geqq 3t$, there is a gap $m+t$ from $4m+6t$ to $5m+7t$, which is less than all later entries. Thus $3+E_3(5m+7t-1)=m+t+2 \equiv \alpha$. If $t=1$, then $m \geqq 3$, $\alpha=L+1$, for $L=m+2$ of Theorem 4. If $t>1$, write $m=ut+v$, $0 \leqq v<t$, whence $u \geqq 3$. Since $m>u$, $\alpha>L$.

Next, let $2t<m<3t$, whence $u=2$, $t>1$. There is a gap $2m-2t$ from $4m+6t$ to $6m+4t$, and $3+E_3=2m-2t+2>L=t+4$.

Next, let $t<m \leqq 2t$. There is a gap m from $21m+12t$ to $v=22m+12t$. Consider $\beta=3+E_3(v-1)=m+2$. If $t=1$, then $m=2$, and $\beta=4=4+e_4$. Let $t>1$. If $m>t+1$, then $\beta \geqq L=t+4$. There remains the case $m=t+1$. Then $4m \leqq 7t$ and $\beta=4+e_4$ by Theorem 7.

When $m=t$, $f(x)$ is the product of t by the triangular number $\frac{1}{2}(x^2+x)$. Hence every multiple of t is a sum of three values of $f(x)$, so that $e_3=t-1$.

The methods of this paper do not apply directly to the remaining case $m<t$. An extension of the above table of sums by three was used to verify that $e_3(A) \leqq t-1$ for $A \leqq 400m+50t$.

12. *Comparison of $2+e_2$ with $s+e_s$.*

First, let $m \geqq t$. There is a gap from $39m+12t$ to $a=42m+13t$ in the table of sums of two values of $f(x)$, and $\sigma=2+E_2(a-1)=3m+t+1$. In all cases, $\sigma \geqq L$ of Theorem 4. Also, $\sigma>t+3$ in the last part of Theorem 8. If $m=t$, $\sigma>3+e_3=t+2$ of Theorem 9.

Second, let $2m \geqq t>m$. There is a gap from $3m+4t$ to $4m+5t$, whence $\tau=2+E_2=m+t+1$. If $m \geqq 2$, $\tau \geqq t+3$ of Theorem 8. In

the remaining case $m=1$, $t=2$, there is a gap 4 from $27m+11t=29m+10t=49$ to $31m+11t=53$, and $2+E_2(52)=5=t+3$.

Third, let $3m \geqq t > 2m$. There is a gap from $8t+21m$ to $9t+22m$, and $2+E_2=t+m+1 \geqq t+3$ unless $m=1$, whence $t=3$. In the latter case, there is a gap 5 from $70=11t+37m=12t+34m$ to $75=10t+45m=12t+39m=13t+36m$, and $2+E_2(74)=6=t+3$.

Fourth, let $4m \geqq t > 3m$. There is a gap from $9t+18m$ to $9t+22m$, and $2+E_2=4m+1 \geqq t+3$ unless $4m=t$ or $t+1$. When $t=4m$, the (first) gap $6m$ is from $394m$ to $400m$, and $2+E_2=6m+1 \geqq t+3$. Finally, let $4m=t+1$. Since $4m \geqq t$ and $7m \leqq 2t$, there is a gap $5m$ from $16t+60m$ to $16t+65m$. Here $2+E_2=5m+1 \geqq t+3$.

Fifth, let $5m \geqq t > 4m$. There is a gap $5m$ from $10t+24m$ to $10t+29m$, and $5m+1 \geqq t+3$ unless $5m=t$ or $t+1$. When $t=5m$, the (first) gap $8m$ is from $962m$ to $970m$. Finally, let $5m=t+1$. Since $5m \geqq t$, $9m \leqq 2t$, there is a gap $6m$ from $17t+70m$ to $17t+76m$, and $6m+1 \geqq t+3$.

Sixth, let $6m \geqq t > 5m$. Write $t=5m+g$, $0<g \leqq m$. If $m \leqq 2g$, there is a gap $6m$ from $\alpha=15t+55m$ to $\beta=15t+61m$, and $6m+1 \geqq t+3$ unless $6m=t$ or $t+1$. If $m>2g$, $\gamma=13t+66m$ is the only entry between α and β, so that there is a gap $2t-5m=5m+2g$ from γ to β. It is $\geqq t+2$ unless $g=1$. Hence there remain only three cases:

$$(34) \qquad t=6m; \quad t=6m-1,\ m>1; \quad t=5m+1,\ m>2.$$

When $t=6m$, the first gap $>6m$ is from $952m=41t+706m$ to $960m=49t+666m$. In the last two cases (34), we have found a gap $t+1$, whence $2+E_2=t+2$. That this is the value of $3+e_3$ is plausible * by the last part of Theorem 9.

Seventh, let $7m > t > 6m$. Write $t=6m+g$. The only possible entries between $\alpha=32t+249m$ and $\beta=32t+256m$ are $\gamma=29t+268m$, $\delta=25t+300m$, $\epsilon=28t+282m$. If $2m \geqq 7g$ there is a gap $6m+3g$

* If we do not admit this, we need gaps $\geqq t+2$. There is a gap $6m-3g$ from $17t+70m$ to $14t+91m$; in the third case (34), it is $\geqq t+2$ if $m \geqq 6$. For $m=3$, $t=16$, the first gap 18 is from $1664=32t+384m$ to $1682=29t+406m=38t+358m$. For $m=4$, $t=21$, the first gap $\geqq 23$ is from $3516=40t+669m$ to $3540=48t+633m$. For $m=5$, $t=26$, the gaps $>t$ below 7000 are $28=4413-4385$, $29=4847-4818$, $41=4899-4858$, $36=5397-5361$, $33=5726-5693$, $30=5799-5769$, $30=6503-6473$, $31=6582-6551$.

$\geqq t+2$ from γ to β. If $2m<7g$, $m\geqq 3g$, there is a gap $8m-4g\geqq t+1$ from γ to δ. If $m<3g$, there is a gap $z=9m-7g$ from α to δ, and $z>t$ if $3m>8g$. Next, the only possible entries between $a=36t+315m$ and $b=36t+322m$ are $c=32t+340m$ and $d=30t+354m$. If $m\leqq 2g$, there is a gap $7m>t$ from a to b. If $m>2g$, there is a gap $w=4m+6g$ from d to b, and $w>t$ if $5g>2m$. We have now found a gap $>t$ except when both $3m\leqq 8g$ and $5g\leqq 2m$. These imply $2g\leqq m\leqq 6g$, which are the conditions for a gap $6m+3g>t$ from $39t+601m$ to $42t+589m$.

THEOREM 10. *At least when $t<7m$, $2+e_2$ is not less than the minimum $s+e_s$ for $s\geqq 3$.*

PART II. THE CASE $k>0$, $m\geqq 2t$, §§ 13-27.

13. THEOREM 11. *If $k>0$, $s\geqq 5$, $m\geqq (s+2)t$, then $e_s=m-(s+1)t-1$.*

No smaller value of e_s will serve since no number between st and $m-t$ is a sum of s values of $q(x)=f(x-k)$ for integers $x\geqq 0$. To show that $0=f(0)$ and $t=f(1)$ are the only such values $<m-t=f(-1)$, we recall from § 2 that

$$f(-p)>f(-1),\quad f(p)>f(1)\quad \text{if } p>1. \tag{35}$$

If N is any integer $\geqq 0$, it remains to prove the existence of s integers $x\geqq 0$ such that

$$N=r+\sum_s q(x),\quad 0\leqq r\leqq e_s. \tag{36}$$

We shall first prove this for $k=1$. Then the summands are chosen from $f(-1)$ and $f(z)$, $z\geqq 0$. Since these are values of $f(y)$ when $y\geqq -k$, (36) will hold also for $k>1$. Write $N=A+R$, where

$$A=\sum_4 q(x),\quad R=r+\sum_{s-4} q(x),\quad 0\leqq r\leqq e_s. \tag{37}$$

We can find a value of R congruent to any assigned integer modulo m such that $R\leqq \rho=11m-t-1$.

If $s=5$, we employ $m-t$ and the first six values

$$0,\ t,\ \kappa=m+2t,\ \lambda=3m+3t,\ \mu=6m+4t,\ \nu=10m+5t \tag{38}$$

of $f(x)$. To $10m+5t$ we add $0, 1, \cdots, e_5=m-6t-1$. To the remaining six we add only $0, 1, \cdots, t-1\leqq e_5$. We get m integers $R\leqq\rho$ which are congruent modulo m to the consecutive integers from $-t$ to $m-t-1$ inclusive.

If $s \geqq 6$, we add $0, 1, \cdots, t-1 \leqq e_s$ to the $s-3$ sums by $s-4$ of 0 and t, and also to $(s-5)t+\kappa$, $(s-6)t+2\kappa$, $2(m-t)$, and $m-t$. But we add $0, 1, \cdots, e_s$ to $\phi = (s-6)t+\kappa+\lambda$. We obtain m integers R which are congruent modulo m to the consecutive integers from $-2t$ to $m-2t-1$. The maximum such R is $\phi + e_s = 5m-2t-1 < \rho$.

We shall treat the first equation (37) by the method of § 4 with $e_4 = 0$. By (4) with $k = 1$, and (7),

$$n = 2t-3m, \quad C = m-t, \tag{39}$$

$$D = A-4C = m(a-3b)/2 + tb. \tag{40}$$

Let $d = 2$. Then there is an odd value of b between the limits (10) and (12) on b. We proved that we can choose $R \leqq \rho$ so that $A = N-R$ is congruent modulo m to any assigned integer. As the latter we take $4C+tb$. Then (40) determines an integer $(a-3b)/2$ and hence an odd integer a, which is positive since $4a > b^2$. By (16), $P = 2t-m < 0$. Hence it remains only to satisfy the minor conditions in § 4. In (9), (19), and (14), we replace n by its value (39) and D by $A-4m+4t$, and get

$$U = 24mA - 63m^2 + 12mt + 36t^2 > 0 \quad \text{if } A \geqq 3m, \tag{41}$$

$$16V-U = 8mA + 79m^2 - 76mt + 28t^2 - 32me_4 > 0 \quad \text{if } A \geqq 0, \tag{42}$$

$$3A \geqq 11m - 4t - 4t^2/m. \tag{43}$$

The last holds if $A \geqq 4m-3t$. Hence (36) holds for every $N \geqq 15m-4t-1$.

We next verify (36) when $N \leqq 15m+t-1$. We employ the abbreviations (38) and $\delta = m-t$. Consider a sum Σ of p of $\delta, \kappa, \lambda, \mu, \nu$, where $p \leqq 5$. To Σ add the sums by $s-p$ of 0 and t. Hence $\Sigma, \Sigma+t, \Sigma+2t, \cdots, \sigma = \Sigma+(s-p)t$ are sums by s of values of $q(x)$. To each we add $0, 1, \cdots, t-1 \leqq e_s$. We get the consecutive integers from Σ to $M = \Sigma+(s+1-p)t-1$ inclusive. We readily verify the following table:

Σ	$M+1$	Σ	$M+1$
0	$(s+1)t^*$	$\delta+\lambda$	$4m+(s+1)t$
δ	$m+(s-1)t$	$\kappa+\lambda$	$4m+(s+4)t^*$
κ	$m+(s+2)t^*$	$2\delta+\lambda$	$5m+(s-1)t$
2δ	$2m+(s-3)t$	$\delta+\kappa+\lambda$	$5m+(s+2)t$
$\delta+\kappa$	$2m+st$	$2\kappa+\lambda$	$5m+(s+5)t^*$
2κ	$2m+(s+3)t^*$	$2\delta+\kappa+\lambda$	$6m+st$
$2\delta+\kappa$	$3m+(s-2)t$	μ	$6m+(s+4)t$
λ	$3m+(s+3)t$	2λ	$6m+(s+5)t^*$
3κ	$3m+(s+4)t^*$	$\delta+\mu$	$7m+(s+2)t$

Σ	$M+1$	Σ	$M+1$
$\kappa+\mu$	$7m+(s+5)t^*$	$\delta+\nu$	$11m+(s+3)t$
$2\delta+\mu$	$8m+st$	$\kappa+\nu$	$11m+(s+6)t$
$\delta+\kappa+\mu$	$8m+(s+3)t$	$2\kappa+\lambda+\mu$	$11m+(s+8)t^*$
$2\kappa+\mu$	$8m+(s+6)t^*$	$\delta+\kappa+\nu$	$12m+(s+4)t$
$2\delta+\kappa+\mu$	$9m+(s+1)t$	2μ	$12m+(s+7)t^*$
$2\delta+\kappa+2\lambda$	$9m+(s+2)t$	$2\delta+\kappa+\nu$	$13m+(s+2)t$
$\delta+2\kappa+\mu$	$9m+(s+4)t$	$\delta+2\mu$	$13m+(s+5)t$
3λ	$9m+(s+7)t^*$	$\kappa+2\mu$	$13m+(s+8)t^*$
ν	$10m+(s+5)t$	$2\delta+2\mu$	$14m+(s+3)t$
$\kappa+\lambda+\mu$	$10m+(s+7)t^*$		

Except for the cases marked *, $M+1$ is $\geqq$ the Σ in the next line, since $s\geqq 5$. For the cases marked *, $m-(s+2)t+M+1$ is $\geqq$ the Σ in the next line. This addition to $M+1$ is permissible since we may add e_s instead of $t-1$ to σ. Making the same addition to the final entry, we see that (36) holds for every $N\leqq 15m+t-1$. This completes the proof of Theorem 11.

14. THEOREM 12. *If $k>0$, $s\geqq 5$, $2t\leqq m<(s+2)t$, then $e_s=t-1$.*

No smaller value of e_s will serve since (35) and $f(-1)=m-t\geqq t$ show that no value of $q(x)$ lies between 0 and t.

We can find a small value of R in (37) which is congruent modulo m to any assigned integer.

If $s=5$, we add $0, 1, \cdots, t-1$ to $m-t$ and the six numbers (38). We get $7t$ integers R congruent modulo m to the consecutive integers from $-t$ to $6t-1$ inclusive. They include a complete set of residues modulo $m<7t$. The maximum such R is $10m+6t-1$. If $s\geqq 6$, we proceed as in § 13, except that we add only $0, 1, \cdots, t-1$ to ϕ. We get $(s+2)t>m$ integers R congruent modulo m to the consecutive integers from $-2t$ to $st-1$ inclusive. The maximum such R is $4m+st-1$.

It suffices to prove (36) for $k=1$. We study (37_1) by the method of § 4 with $e_4=0$, $d=2$. We have (39)—(43). Also F is given by (24). Since n is negative, $F>0$ for every D. Since $m\geqq 2t$, (43) holds if $A\geqq 4m-2t$. Adding to this the above maximum of R, we conclude that (36) holds if $N\geqq$ the larger of $14m+4t-1$ and $8m+(s-2)t-1$.

For the remaining values of N we proceed as at the end of § 13. Each entry $M+1$ marked by a * in the table exceeds the Σ in the next line since $(s+2)t>m$.

15. THEOREM 13. *Let $k>0$, $m\geqq 2t$, and write L for the least $s+e_s$ for $s\geqq 5$. When $t=1$, $L=m-2$ if $m\geqq 7$, $L=5$ if $2\leqq m<7$. When*

$t > 1$, write $m = ut + v$, $0 \leqq v < t$, whence $u \geqq 2$. Then $L = u + t - 3$ if $u \geqq 7$, $v = 0$; $L = u + t - 2$ if $u \geqq 7$, $v \geqq 1$; $L = t + 4$ if $2 \leqq u \leqq 6$.

First, let $m \geqq (s+2)t$, $s \geqq 5$. By Theorem 11 of § 13, the least $s + e_s$ is $m - 2$ if $t = 1$, $m \geqq 7$; but is $u + v + t - 3$ *if* $t > 1$, $u \geqq 7$, since $u \geqq s + 2 \geqq 7$ and the maximum s is $u - 2$.

Second, let $2t \leqq m < (s+2)t$, $s \geqq 5$. By Theorem 12, $e_s = t - 1$. If $t = 1$, the least $s + e_s = s$ is 5 if $2 \leqq m < 7$, but is $m - 1$ if $m \geqq 7$. If $t > 1$, then $s \geqq u - 1$ and the least $s + e_s$ is $u - 1 + t - 1$ if $u \geqq 7$, but is $5 + t - 1$ if $u \leqq 6$.

Note that $m < 7t$ if and only if $u \leqq 6$. Then $4 + e_4 \geqq L$ if and only if $e_4 \geqq t$ and hence if and only if there is a gap $> t$ in table T_k of sums by four of values of $f(x-k)$ for $x \geqq 0$.

16. *Case* $s = 4$, $t = 1$. First, let $m \geqq 7$. Then 0 and 1 are the only values $< m - 1$ of $f(x-k)$. Hence to obtain the sum $m - 2$, we must employ $m - 2$ values 1. Thus $E_4(m-2) = m - 6$, and $m - 2$ is the minimum $s + e_s$ for $s \geqq 4$. Second, let $m < 7$. We employ the following results proved in *Bulletin*, II. If $k = 1$, $4 + e_4$ is 6 if $m = 6$, 5 if $m = 4$ or 5, 4 if $m = 3$. If $k \geqq 2$, $4 + e_4$ is 5 if $m = 6$, 4 if $m = 3$, 4, or 5. When $m = 2$, $f(x) = x^2$ and $e_4 = 0$. This proves

THEOREM 14. *If $k > 0$, $t = 1$, the minimum of $s + e_s$ for $s \geqq 4$ is $m - 2$ when $m \geqq 7$, 5 when $m = 6$, 4 when $m = 2$ or 3, but when $m = 4$ or 5, it is 5 if $k = 1$, 4 if $k \geqq 2$.*

17. *Case* $s = 4$, $t > 1$, $k = 1$, $m \geqq 2t$. A table T_1 of all sums by four of values of $f(x-1)$ for integers $x \geqq 0$ is derived from Table II of *Bulletin*, II, by multiplying the terms free of m by t. We employ the notations of Theorem 13. If $u \geqq 6$, there is a gap $m - 5t$ from $4t$ to $m - t$, whence

$$g \equiv 4 + E_4(m - t - 1) = ut + v - 5t + 3.$$

If $u \geqq 7$, $v = 0$, $g - L = (u-6)(t-1) > 0$. If $u \geqq 7$, $v \geqq 1$, $g - L = (u-6)(t-1) + v - 1 > 0$. If $u = 6$, $g - L = v - 1 \geqq 0$ unless $v = 0$. But if $m = 6t$, there is a gap $2t > t$ from $5m + 2t = 4m + 8t$ to $5m + 4t < 6m$. If $u = 5$, there is a gap $2t - v > t$ from $4m - 4t$ to $3m + 3t$. If $u = 4$, there is a gap $t + v$ from $2m + 6t$ to $3m + 3t$. But if $u = 4$, $v = 0$, there is a gap $2t$ from $5m + 5t$ to $5m + 7t$. If $u = 3$, there is a gap $t + v$ from $5m + 5t$ to $6m + 3t$; if also $v = 0$, we apply the following with $h = 3$.

If $m = ht$, $f(x) = tp(x)$, where $p(x)$ is a polygonal number of order $h+2$. For $h = 2$ or 3, every integer $\geqq 0$ is a sum of four values of $p(x-1)$ by *Bulletin*, II. Then every multiple of t is a sum of four values of $f(x-1)$. Hence $e_4 = t-1$.

This covers the case $u = 2$, $v = 0$. There remains only the case $2t < m < 3t$.

18. THEOREM 15. *If $k = 1$, $t > 1$, $2m \geqq 5t$, $m < 3t$, then $e_4 = t-1$.*

No smaller e_4 will serve since t is the least value > 0 of $f(x-1)$. Here n and C have the values (39). By (7),

$$D = A - 4C = mq + bt + r, \quad q = \tfrac{1}{2}(a - 3b). \tag{44}$$

Assign to b the odd values β, $\beta+6$, $\beta+4$, $\beta+2$, and to r the values $0, 1, \cdots, t-1$. The residues modulo m of $bt+r$ are βt plus

$$0, \cdots, t-1;\ 6t-2m, \cdots, 7t-2m-1;\ 4t-m, \cdots, 5t-m-1;$$
$$2t, \cdots, 3t-1.$$

Here $0 < 6t-2m \leqq t$, $0 < 4t-m < 7t-2m$, $2t < 5t-m$. Hence these four sets include the consecutive integers from 0 to $3t-1 > m-1$. Hence $d = 8$. We have U in (41) and

$$V = 2mA + 4t^2 - 6tm + m^2 + 2m, \quad P = 2t + 11m,$$
$$F = m^2\omega^2 - 216t^3m - 2664t^2m^2 - 9702tm^3 - 8712m^4 + 24t^2m + 264tm^2 + 726m^3,$$

where $\omega = A - 52t - 96m + 4$. Multiply $2m \geqq 5t$ by $44t^2m$, $3t > m$ by $9702tm^2$, and $9t^2 > m^2$ by $8712m^2$. Hence $F > 0$ if $m^2\omega^2 \geqq ht^2m^2$,

$$h = 88 + 2664 + 29106 + 78408 = 110266 < 333.^2$$

Hence $F > 0$ if $\omega \geqq 333t$ and therefore if

$$A \geqq 69(2m - 5t) + 385t + 96m = 234m + 40t.$$

Then the reduced minor conditions in § 4 are satisfied. For, (14) holds if $3D \geqq 8t - m - 4t^2/m$ and hence if $D \geqq 2t$, $A \geqq 4m - 2t$. Also, (20) hold if $A \geqq 4m - 4$.

The gaps in table T_1 are all $\leqq t$ down to $d = 27m + 9t$, and from $b = 25m + 15t$ to the limit $54m + 19t$, when $m < 3t$. Now $6t - 2m$ is positive and $\leqq t$; adding it to d we get b.

We employ table T_0, cited in § 9, of sums by four of values of $f(x)$. To it we annex the sums

$$m-t+2(21m+7t)+45m+10t=88m+23t,$$
$$m-t+m+2t+10m+5t+105m+15t=117m+21t=\sigma$$

of four values of $f(x-1)$. Using only $m<3t$, we see that all gaps are now $\leqq t$ from $54m+19t$ to either $112m+32t$ or σ and (§ 10) from $\tau=115m+27t$ to $234m+45t$. But τ is the sum of σ and $6t-2m\leqq t$. This completes the proof of Theorem 15, which covers the case $2v\geqq t$ in the notation of Theorem 16.

19. THEOREM 16. *If $k=1$, $m=2t+v$, $0<2v<t$, then $e_4\geqq t$ except when $8v\leqq t$ or $4v=t$, and then e_4 is probably $t-1$, with certainty that $e_4(A)=t-1$ if $A\leqq 400m+58t$.*

In § 18 we cited the entries b, d, of table T_1. They may be written as $b=65t+25v$, $d=63t+27v$. All entries involving gm, where $g\geqq 29$, exceed b. Those involving lm, where $l\leqq 23$, are $<d$. Also, $24m+15t<d$. From the entry $e=24m+16t$ and the entries involving $25m$, $26m$, $27m$, $28m$, we eliminate m and see that $e=64t+24v$ and $f=64t+28v$ are the only possible entries between d and b.

If $3v\geqq t$, then $e\leqq d$, $f\geqq b$, and there is a gap $2t-2v>t$ from d to b. Hence let $3v<t$. Then $e-d=b-f=t-3v$, and there is a gap $4v$ from e to f.

There remains only the case $4v\leqq t$. As just noted, the gaps from d to e, e to f, and f to b are all $\leqq t$. Then by § 18, the gaps in T_1 are $\leqq t$ to $\alpha=112m+32t=256t+112v$, and from $\beta=113m+31t=257t+113v$ to τ and thence to $234m+45t$ and (§ 10) on to $400m+58t$. We employ the following extension of table T_1:

$113m+17, 20, 21, 23—29, 31, 32$; $114m+19, 22—27, 29—32$; $115m+20—22, 24, 25, 27—32$; $116m+19, 20, 22, 23, 25—32$; $117m+18, 21, 23—29, 31, 32$; $118m+22—32$; $119m+22—25, 27—32$; $120m+16—19, 21—33$; $121m+15—21, 23—29, 31—33$; $122m+14, 15, 17—23, 25—33$; $123m+13, \cdots$,

in which the terms free of m are to be multiplied by t.

The only possible entries between α and β are

$$\gamma=117m+21t=255t+117v,\quad \delta=120m+16t=256t+120v.$$

In fact,

$$113m+29t=255t+113v<\alpha,\quad 114m+27t=255t+114v<\alpha,$$
$$114m+29t=257t+114v>\beta,\quad 115m+27t=257t+115v>\beta,$$
$$115m+25t=255t+115v<\alpha,\quad 116m+23t=255t+116v\leqq\alpha,$$

$$116m+25t=257t+116v>\beta, \quad 117m+18t=252t+117v<\alpha,$$
$$117m+23t=257t+117v>\beta, \quad 118m+22t=258t+118v>\beta,$$
$$120m+17t=257t+120v>\beta, \quad 121m+15t=257t+121v>\beta,$$
$$122m+14t=258t+122v>\beta, \quad 123m+13t=259t+123v>\beta.$$

Each entry $gm+ht$ with $g\geqq 124$ exceeds β since $h>13$.

First, let $5v\leqq t$. We drop $\gamma\leqq\alpha$. If $7v\geqq t$, we drop also $\delta\geqq\beta$ and have a gap $t+v>t$ from α to β. Hence let $7v<t$. Then there is a gap $8v$ from α to δ, and a gap $t-7v$ from δ to β. There remains only the case $8v\leqq t$. If we attempt to prove that $e_4=t-1$, we see that $d=20$ by either of our methods and that there remain numbers far beyond our tables.

Second, let $5v>t$, $4v\leqq t$. We drop $\delta>\beta$. There is a gap $5v-t$ from α to γ and a gap $w=2t-4v$ from γ to β. Since $w>t$ if $t>4v$, there remains only the case $t=4v$. If we then attempt to prove that $e_4=t-1$, we may take $v=1$ and see that $d=12$ by either of our methods. We find that $F>0$ if $A\geqq 6098$. The limit at the end of Theorem 16 is now 3832.

Combining Theorems 15 and 16 with the results in § 17, we get

THEOREM 17. *If* $k=1$, $t>1$, $m\geqq 2t$, *the minimum of* $s+e_s$ *for* $s\geqq 4$ *is* L *of Theorem* 13 *if* $m>3t$ *or if* $m=2t+v$, $0<2v<t$, *and neither* $8v\leqq t$ *nor* $4v=t$. *But it is* $4+e_4=t+3=L-1$ *if* $m=2t$ *or* $m=3t$ *or* $m=2t+v$, $0<v<t\leqq 2v$, *and also up to* $400m+58t$ *if either* $8v\leqq t$ *or* $4v=t$.

20. If $s=4$, $t>1$, $k\geqq 2$, $m>6t$, there is a gap from $4t$ to $m-t$ and the first part of § 17 shows that $4+e_4\geqq L$.

21. *The Case* $s=4$, $k=2$, $t>1$, $m\geqq 2t$. If $5t\leqq m\leqq 6t$, there is a gap from $9m+5t$ to $10m+t$ in T_2 and the gap exceeds t unless $m=5t$. In that case, $e_4=t-1$ by the end of § 17 and *Bulletin*, II, Theorem 4.

Let $m=4t+v$, $0\leqq v<t$. If $3v>2t$, there is a gap $2t-v>t$ from $28m+2t$ to $27m+8t$. If $3v\leqq 2t$, there is a gap $2v$ from $25m+16t$ to $27m+8t$. There remains only the case $2v\leqq t$. We shall prove that $e_4=t-1$. The use of only odd values of b leads to $d=18$ and hence to a limit for A beyond our tables. That we may take $d=9$ is shown by the following method to be applied frequently in the sequel.

For any k, (4) and (7) give

$$D=mH+tb+r, \quad H=\tfrac{1}{2}[a-(1+2k)b]. \tag{45}$$

LEMMA 4. *If* $b \equiv 2 \pmod 4$, *the choice* $D \equiv tb + r \pmod{2m}$ *yields an integer* $a \equiv 2 \pmod 4$ *which satisfies* (45). *The same is true of the choice* $D \equiv tb + r + m \pmod{2m}$ *if* $b \equiv 0 \pmod 4$. *If* b *is odd, either of these two choices yields an odd integer* a *satisfying* (45).

In the first case, H is even, whence $a \equiv b \equiv 2 \pmod 4$. In the second case, $m \equiv mH \pmod{2m}$ and H is odd, whence $a - b \equiv 2$, $a \equiv 2 \pmod 4$. If b is odd, either choice makes $D \equiv tb + r \pmod m$, whence H is an integer and a is odd. In all these cases we may apply Lemma 1 or 2.

We seek the smallest set * of consecutive values β, $\beta + 1, \cdots$ of b such that, when r takes its values $0, 1, \cdots, t - 1$, the values of D computed by Lemma 4 shall yield a complete set of residues modulo $2m$. Then every integer D can be given the form (45) by use of permissible values of a and b for which Lemmas 1 and 2 are applicable.

TABLE T_2.

All Sums by Four of Values of $f(x - 2)$.

(terms free of m are to be multiplied by t)

0—4; m—1, 0—5; $2m$—2, —1, 0—6; $3m$—3—1, 0, 1, 3—7; $4m$—4—1, 0—8; $5m$—4, —3, —1, 0—5, 7, 8; $6m$—5—2, 0—9; $7m$—5, —4, —2, —1, 0, 1, 3—9; $8m$—6, —3, —1, 0, 2—10; $9m$—6, —5, —1, 0—5, 7—10; $10m$—7, —4, —2, 1—11; $11m$+0, 3—9, 11; $12m$—8, —3, 0—12; $13m$—1, 2—5, 7—12; $14m$+2, 3, 5—12; $15m$—2, 1, 3, 4, 6—9, 11—13; $16m$+1, 2, 5—13; $17m$+0, 3—5, 7—13; $18m$+3—7, 9—14; $19m$—1, 3, 4, 6—9, 11—14; $20m$+2, 5—14; $21m$+2, 3, 7—13, 15; $22m$+1, 4—15; $23m$+5, 6, 8, 9, 11—15; $24m$+0, 4—16; $25m$+4, 5, 7—13, 15, 16; $26m$+3, 6, 8—11, 13—16; $27m$+3, 4, 6, 8, 9, 11—16; $28m$+2, 5, 7—17; $29m$+7—13, 15—17; $30m$+1, 6, 7, 9—17; $31m$+5—8, 11—17; $32m$+5, 6, 8—18; $33m$+4, 7, 9—13, 15—18; $34m$+4, 5, 9—18; $35m$+3, 6, 8, 9, 11—17; $36m$+8—19; $37m$+2, 7—13, 15—19; $38m$+7—15, 17—19; $39m$+6—9, 11—17, 19; $40m$+6—20; $41m$+5, 8, 10—13, 15—20; $42m$+5, 6, 9—11, 13—20; $43m$+4, 7, 9, 12—17, 19, 20; $44m$+9, 11—20; $45m$+3, 8, 10—13, 15—21; $46m$+9—21; $47m$+8, 9, 11—17, 19—21; $48m$+7—10, 12—21; $49m$+7, 8, 10—13, 15—21; $50m$+6, 9, 11, 12, 14—22; $51m$+6, 7, 11—17, 19—22; $52m$+5, 8, 10, 13—22; $53m$+12, 13, 15—21; $54m$+4, 9, 12—14, 17—19, 21, 22; $55m$+11—17, 19—23; $56m$+10—23; $57m$+9—13, 15—21, 23; $58m$+8—11, 13—23; $59m$+8, 9, 11—17, 19—23; $60m$+7, 10, 12—24; $61m$+7, 8, 11—13, 15—21, 23, 24; $62m$+6, 9, 11, 12, 14—24; $63m$+12—17, 19—24; $64m$+5, 10, 13—24; $65m$+12, 15—21, 23, 24; $66m$+12—25; $67m$+11—17, 19—25; $68m$+10, 11, 13—25; $69m$+9—13, 15—21, 23—25; $70m$+9, 10, 12—15, 17—19, 21—25; $71m$+8, 11—14, 16, 17, 19—25; $72m$+8, 9, 13—26; $73m$+7, 10, 12, 15—21, 23—26; $74m$+14, 16—23,

* This was true for a single b in the simple case of (II), § 9.

25, 26; $75m+6$, 11, 14—17, 19—25; $76m+13$, 15—26; $77m+15$—21, 23—26; $78m+13$—27; $79m+12$—17, 19—25, 27; $80m+11$, 12, 14, 15, 17—27; $81m+10$—13, 16—21, 23—27; $82m+10$, 11, 13—27; $83m+9$, 12, 14—17, 19—25, 27; $84m+9$, 10, 14—28; $85m+8$, 11, 13, 15—21, 23—28; $86m+15$—18, 21—28; $87m+7$, 12, 14—17, 19—25, 27, 28; $88m+14$, 17—28; $89m+15$, 17, 18, 20, 21, 23—28; $90m+13$, 16—23, 25—28; $91m+14$—17, 19—25, 27—29; $92m+13$—29; $93m+12$, 13, 15, 16, 18—21, 23—29; $94m+11$—14, 17—29; $95m+11$, 12, 14—17, 19—25, 27—29; $96m+10$, 13, 15—18, 21—29; $97m+10$, 11, 15, 16, 18—21, 23—29; $98m+9$, 12, 14, 17—27, 29, 30; $99m+15$, 16, 19—25, 27—30; $100m+8$, 13, 16—30; $101m+15$, 18—21, 23—29; $102m+18$, 19, 21—30; $103m+14$, 17, 19—25, 27—30; $104m+17$—19, 21—30; $105m+15$—21, 23—29, 31; $106m+14$—23, 25—31; $107m+13$—17, 19—25, 27—31; $108m+12$—15, 17—22, 24—31; $109m+12$, 13, 15—21, 23—29, 31; $110m+11$, 14, 16—31; $111m+11$, 12, 16, 17, 19—25, 27—31; $112m+10$, 13, 15—32; $113m+17$, 20, 21, 23—29, 31, 32; $114m+9$, 14, 17—27, 29—32; $115m+16$, 19—25, 27—32; $116m+18$—23, 25—32; $117m+15$, 18, 20, 21, 23—29, 31, 32; $118m+17$—19, 22—32; $119m+17$, 20, 22—25, 27—32; $120m+16$—19, 21—33; $121m+15$—21, 23—29, 31—33; $122m+14$, 15, 17—23, 25—33; $123m+13$—16, 19—25, 27—33; $124m+13$, 14, 16—19, 21—33; $125m+12$, 15, 17, 18, 20, 21, 23—29, 31—33; $126m+12$, 13, 17, 18, 20—31, 33; $127m+11$, 14, 16, 19—25, 27—33; $128m+18$, 20—34; $129m+10$, 15, 18—21, 23—29, 31—34; $130m+17$, 19—27, 29—34; $131m+20$, 21, 23—25, 27—33; $132m+16$, 18, 19, 21—34; $133m+19$, 20, 23—29, 31—34; $134m+18$, 21—34; $135m+21$—25, 27—33; $136m+17$—35; $137m+16$—21, 23—29, 31—35; $138m+15$, 16, 18—22, 25—35; $139m+14$—17, 19—25, 27—33, 35; $140m+14$, 15, 17—24, 26, 27, 29—35; $141m+13$, 16, 18, 19, 21, 23—29, 31—35; $142m+13$, 14, 18—31, 33—35; $143m+12$, 15, 17, 20—25, 27—33, 35; $144m+19$, 21—36.

Write $[j]$ for the set $j, j+1, \cdots, j+t-1$.

(I) $\beta \equiv 1 \pmod 4$. For $b = \beta,\ \beta+2,\ \beta+4,\ \beta+6,\ \beta+8$, the values of D computed by Lemma 4 are congruent modulo $2m$ to the sum of $t\beta$ and

$$(46) \qquad [0], [m] \quad [2t], [2t+m] \quad [4t], [4t+m] \quad [6t], [6t+m] \quad [8t], [8t+m].$$

For $b = \beta+1,\ \beta+3,\ \beta+5,\ \beta+7$, we get

$$(47) \qquad [t] \quad [3t+m] \quad [5t] \quad [7t+m].$$

In the present problem, $4t \leqq m \leqq 5t$. We subtract the modulus $2m$ from $7t+m$ and $8t+m$. The sets

$$[0]\ [t]\ [2t]\ [7t-m]\ [8t-m]\ [4t]\ [5t]\ [6t]\ [2t+m]\ [3t+m]\ [4t+m]$$

overlap and are arranged in ascending order, since the first number of any set is $\leqq 1$ plus the last number of the preceding set. The largest number of the final set is $5t+m-1 \geqq 2m-1$. Hence the sets include a complete set of residues modulo $2m$.

(II) $\beta \equiv 3 \pmod 4$. We again have (46). But (47) is replaced by

$$[t+m] \quad [3t] \quad [5t+m] \quad [7t]. \tag{48}$$

The following are overlapping sets in ascending order:

$$[0]\ [5t-m]\ [6t-m]\ [2t]\ [3t]\ [4t]\ [m]\ [t+m]\ [2t+m]\ [7t]\ [8t].$$

The largest number of the final set is $9t-1 \geqq 2m-1$ since $2v \leqq t$.

(III) $\beta \equiv 0 \pmod 4$. For $b = \beta+1, \beta+3, \beta+5, \beta+7$, we get

$$[t], [t+m] \quad [3t], [3t+m] \quad [5t], [5t+m] \quad [7t], [7t+m]. \tag{49}$$

For $b = \beta, \beta+2, \beta+4, \beta+6, \beta+8$, we get

$$[m] \quad [2t] \quad [4t+m] \quad [6t] \quad [8t+m]. \tag{50}$$

The set $[4t+m]$ terminates with $5t+m-1 \equiv 5t-m-1 \geqq 0 \pmod{2m}$ and hence contains the residues $0, 1, \cdots, 5t-m-1$. We employ these and the following overlapping sets in ascending order:

$$[5t-m]\ [t]\ [2t]\ [3t]\ [8t-m]\ [m]\ [t+m]\ [6t]\ [7t]\ [3t+m]\ [4t+m].$$

(IV) $\beta \equiv 2 \pmod 4$. We again have (49). But (50) is replaced by

$$[0] \quad [2t+m] \quad [4t] \quad [6t+m] \quad [8t]. \tag{51}$$

Overlapping sets in ascending order are

$$[0]\ [t]\ [6t-m]\ [7t-m]\ [3t]\ [4t]\ [5t]\ [t+m]\ [2t+m]\ [3t+m]\ [8t].$$

Since in all four cases we obtain a complete set of residues modulo $2m$, we have $d=9$. Then

$$n = 2t-5m, \qquad U = 24mD + 36t^2 - 156tm + 153m^2,$$
$$C = 3m-2t, \qquad V = 2mD + 4t^2 - 22tm + 25m^2 + 2m,$$
$$P = 2t+11m, \qquad W = 3mD + 4t^2 - 25tm + 4m^2,$$
$$F = m^2\omega^2 - 240t^3m - 2856t^2m^2 - 9636tm^3 - 6534m^4 + 24t^2m + 264tm^2 + 726m^3,$$

where $\omega = D - 63t - 75m + 4$. Multiply $m \geqq 4t$ by $60t^2m$, $9t \geqq 2m$ by $4818tm^2$, and $81t^2 \geqq 4m^2$ by $1633\frac{1}{2}m^2$. Hence $F > 0$ if $m^2\omega^2 \geqq pt^2m^2$, $p = 60 + 2856 + 43362 + 132314 = 178592 < 423^2$, and hence if $\omega \geqq 423t$. This holds if

$$A \geqq 109(m - 4t) + 12m - 8t + 423t + 63t + 75m = 196m + 42t.$$

The gaps are $\leqq t$ in T_2 to $122m + 33t$ and thence to $196m + 41t$ in T_0 after annexation of

$$\begin{aligned} 164m + 28t &= m - t + 3m + 3t + 55m + 11t + 105m + 15t, \\ 164m + 29t &= m - t + 6m + 4t + 66m + 12t + 91m + 14t, \\ 194m + 33t &= m - t + 28m + 8t + 45m + 10t + 120m + 16t. \end{aligned}$$

Next, let $3t \leqq m < 4t$. To prove that $e_4 = t - 1$, give to b the values β, $\beta + 1$, $\cdots$, $\beta + 7$. We suppress the final pair in (46), and the final entry in both (50) and (51). If $\beta \equiv 1 \pmod 4$, the sets

$$[0] \quad [t] \quad [2t] \quad [6t - m] \quad [7t - m] \quad [4t] \quad [5t] \quad [6t] \quad [3t + m]$$

overlap and are in ascending order. The largest number in the last set is $4t + m - 1 > 2m - 1$. If $\beta \equiv 3 \pmod 4$, the sets

$$[0] \quad [4t - m] \quad [5t - m] \quad [2t] \quad [3t] \quad [4t] \quad [t + m] \quad [2t + m] \quad [6t] \quad [7t]$$

overlap and terminate with $8t - 1 > 2m - 1$. If $\beta \equiv 0 \pmod 4$, the sets

$$[4t - m] \quad [t] \quad [2t] \quad [3t] \quad [m] \quad [t + m] \quad [5t] \quad [6t] \quad [7t]$$

overlap and terminate with $8t - 1 \equiv 8t - 2m - 1 \geqq 4t - m \pmod{2m}$, and hence include a complete set of residues modulo $2m$. If $\beta \equiv 2 \pmod 4$, the sets

$$[0] \quad [t] \quad [5t - m] \quad [6t - m] \quad [3t] \quad [4t] \quad [5t] \quad [2t + m] \quad [3t + m]$$

overlap and terminate with $4t + m - 1 > 2m - 1$. In every case, $d = 8$. Using the above U and V, we get

$$F = m^2\omega^2 - 216t^3m - 2040t^2m^2 - 5238tm^3 - 1944m^4 + 486m^3 + 24t^2m + 216tm^2,$$

where $\omega = D - 56t - 40m + 4$. Multiply $m \geqq 3t$ by $72t^2m$, $4t > m$ by $5238tm^2$, and $16t^2 > m^2$ by $1944m^2$. Hence $F > 0$ if

$$m^2\omega^2 \geqq gt^2m^2, \quad g = 72 + 2040 + 20952 + 31104 = 54168 < 233^2,$$

and therefore if

$$A \geqq 56t + 40m + 233t + 12m - 8t + 82(m - 3t) = 134m + 35t.$$

The gaps are $\leqq t$ to $77m + 26t$ in T_2 and thence to $134m + 34t$ in T_0 after annexation of $m - t + 15m + 6t + 2(36m + 9t)$.

Finally let $m = 2t + v$. In view of §§ 17, 18, it remains to consider only the case $0 < 2v < t$. In T_2, the gaps are $\leqq t$ from $27m + 9t$ to $28m + 17t$. Hence by § 18, the gaps are $\leqq t$ down to $\alpha = 112m + 32t$ and from $\beta = 113m + 31t$ to $400m + 58t$. We employ the entries

$$\epsilon = 117m + 20t = 254t + 117v, \quad \rho = 118m + 18t = 254t + 118v,$$
$$\sigma = 118m + 19t = 255t + 118v, \quad \tau = 119m + 17t = 255t + 119v.$$

(i) Let $5v > t$. Then the gaps from α to β are all $< t$. For, if $5v \geqq 2t$, then $\epsilon - \alpha = 5v - 2t < t$ since $5v < 6v < 3t$, $\rho - \epsilon = v$, $\gamma - \rho = t - v$, $\beta - \gamma = 2t - 4v < t$ since $8v > 5v \geqq 2t$, and these differences are all $\geqq 0$. But if $5v < 2t$, then $\gamma - \alpha = 5v - t$, $\sigma - \gamma = v$, $\beta - \sigma = 2t - 5v$ are all positive and $< t$.

(ii) Let $5v \leqq t$. Then δ, σ, and τ are the only possible entries between α and β in T_2.

First, let $t \leqq 7v$. We drop $\delta \geqq \beta$. If $6v < t$, we drop also $\sigma < \alpha$ and have

$$\tau - \alpha = 7v - t \geqq 0, \quad \beta - \tau = 2t - 6v > t.$$

But if $6v \geqq t$, then $\sigma - \alpha = 6v - t$, $\tau - \sigma = v$, $\beta - \tau = 2t - 6v$ are all $\geqq 0$ and $\leqq t$, whence the gaps from α to β are $\leqq t$.

Second, let $7v < t$. We drop $\sigma < \alpha$, $\tau < \alpha$. Then $\beta - \delta = t - 7v$ and there is a gap $8v$ from α to δ.

In summary, there is a gap $> t$ if and only if $6v < t < 8v$, while the gaps are all $\leqq t$ to $400m + 58t$ if $t \leqq 6v$ or $t \geqq 8v$.

If $v = 1$, then $t = 3, \cdots, 6$ or $\geqq 8$. Is $e_4 = t - 1$?

For $t = 3$, $m = 7$, $e_4 = 2$, either of our methods gives $d = 10$. Then

$$F = m^2D^2 - 6250mD - 2658761 > 0 \text{ if } MD \geqq 6650,$$

and hence if $A \geqq 1010$. But the gaps are $\leqq t$ to $130m + 34t = 1012$.

For $t = 4$, $m = 9$, $e_4 = 3$, either method gives $d = 12$. Then

$$F = m^2D^2 - 41940mD - 9573714 > 0 \text{ if } mD \geqq 42168,$$

and hence if $A \geqq 4762$. This exceeds the limit $400m + 58t = 3832$ of our tables.

For $t = 5$, $m = 11$, or $t = 6$, $m = 13$, or $t = 8$, $m = 17$, the use of odd b's only gives $d = 14$, 16, or 20, respectively, and leads to values of A beyond our tables.

When $v = 2$, let first t be even, $t = 2T$. Then $m = 2M$, $M = 2T + 1$. Then $f(x) = 2F(x)$, where $F(x)$ has the parameters T and M. In certain cases we just proved that its e_4 is $T - 1$, so that every positive integer is a sum of r and four values of $F(x - 2)$ where $r = 0, \cdots, T - 1$. Hence every even integer is a sum of $2r$ and four values of $f(x - 2)$. Thus every odd is a sum of $2r + 1$ and four such values. Hence every integer is a sum of R and four values, where $R \leqq 2T - 1$. Thus $e_4 = t - 1$ for $f(x)$.

For $v = 2$, $t = 5$, $m = 12$, $e_4 = 4$, either method gives $d = 10$. Then

$$F = m^2D^2 - 42000mD - 23263488 > 0 \text{ if } mD \geqq 42547,$$

and hence if $A \geqq 3650 = 300m + 10t$, and the proof is complete.

But for $v = 2$, $t = 7$ or 9, $d = 12$ or 14 and A is too large. For $v = 3$, $t = 7$, $d = 10$.

THEOREM 18. *If $k = 2$, $t > 1$, $m \geqq 2t$, the minimum of $s + e_s$ for $s \geqq 4$ is L of Theorem 13 when $2m > 9t$, $m \neq 5t$, and when $m = 2t + v$, $6v < t < 8v$; but is $4 + e_4 = t + 3 = L - 1$ when $m = 2t$, $m = 5t$, $5t \leqq 2m \leqq 9t$, and for certain (and probably all) cases of $m = 2t + v$ for which either $t \leqq 6v$ or $t \geqq 8v$.*

22. *The Case $s = 4$, $k = 3$, $t > 1$, $m \geqq 2t$.* If $m > 6t$, see § 20. We need the following part of table T_3:

$$38m + 2,\ 5,\ 7 - 15,\ 17 - 19;\ 39m - 2,\ +5 - 9,\ 11 - 17,\ 19;$$
$$40m + 1 - 3,\ 6 - 20;$$

all entries $gm + ht$ with $g \leqq 37$ have $h \leqq 19$; all with $g \geqq 41$ have $h \geqq 1$, except for $46m - 1$. Hence if $5t < m \leqq 6t$, there is a gap $m - 4t > t$ from $38m + 15t$ to $39m + 11t$ in T_3. If $m = 5t$, $e_4 = t - 1$ by § 21.

Let $m = 4t + v$, $0 \leqq v < t$. We shall prove that $e_4 = t - 1$. Since this is true by § 21 if $2v \leqq t$, let $2v > t$, whence $2m > 9t$. If $\beta \equiv 1 \pmod 4$, we employ § 21 unchanged. If $\beta \equiv 3 \pmod 4$ we employ after $[8t]$ the set $[4t + m]$ there ignored; it terminates with $5t + m = 1 > 2m - 1$. If $\beta \equiv 0 \pmod 4$, we must use the new value $b = \beta + 9$, which gives the set $[9t - m]$; we insert it before $[m]$. If $\beta \equiv 2 \pmod 4$, we must use also

$b = \beta + 9$; we place the new set $[9t]$ at the end; it terminates with $10t - 1 > 2m - 1$. In each case we may take $d = 10$. Then

$$F = m^2\omega^2 - 264t^3m - 3024t^2m^2 - 9306tm^3 - 3630m^4 + 24t^2m + 264tm^2 + 726m^3,$$

where $\omega = D - 70t - 51m + 4$. Multiply $2m > 9t$ by $30t^2m$, $5t > m$ by $9306tm^2$, and $25t^2 > m^2$ by $3630m^2$. Hence $F > 0$ if $m^2\omega^2 \geqq lt^2m^2$, $l = 60 + 3024 + 46530 + 90750 = 140364 < 375^2$. But $A = D + 4C$, $C = 6m - 3t$. Hence $F > 0$ if

$$A \geqq 75m + 433t + 44(2m - 9t) = 163m + 37t.$$

To T_2 we annex $27m + 7t = 6m - 3t + 6m + 4t + 15m + 6t$, which is a sum of three and hence of four values of $f(x - 3)$. Hence by § 21 the gaps in T_3 are $\leqq t$ to $163m + 37t$.

Next, let $m = 2t + v$. The only possible numbers between α and β in T_3, but not in T_2, are $120m + 14t$ and $123m + 6t$. They are both $< \alpha$ in the case $6v < t < 8v$ in which there was a gap $> t$ in T_2.

THEOREM 19. *If $k = 3, t > 1$, $m \geqq 2t$, the minimum of $s + e_s$ for $s \geqq 4$ is L of Theorem 13 when $m > 5t$, and when $m = 2t + v$, $6v < t < 8v$; but is $4 + e_4 = t + 3 = L - 1$ when $m = 2t$, $5t \leqq 2m \leqq 10t$, and for certain (and probably all) cases of $m = 2t + v$ for which either $t \leqq 6v$ or $t \geqq 8v$.*

23. THEOREM 20. *In $k = 4, t > 1$, $m \geqq 2t$, the minimum of $s + e_s$ for $s \geqq 4$ is L of Theorem 13 when $2m > 11t$, and when $m = 2t + v$, $6v < t < 8v$. But it is $4 + e_4 = t + 3 = L - 1$ for certain (if not all) cases of $m = 2t + v$, $t \leqq 6v$ or $t \geqq 8v$, for $m = 2t$, for $5t \leqq 2m \leqq 10t$, and probably for $10t < 2m \leqq 11t$ since then $E_4(A) = t - 1$ if $A \leqq 354m + 55t$ and if $A \geqq 689m + 75t$.*

If $m > 6t$, see § 20. If $5t \leqq m \leqq 6t$, there is a gap $2m - 10t$ from $B = 69m + 21t$ to $C = 71m + 11t$ in T_4:

$69m+4$, 7, 9—13, 15—21, 23—25; $70m+3$, 8—10, 12—15, 17—19, 21—25; $71m$—1, +3—8, 11—17, 19—25; $72m+3$, 6, 8—26; $73m+2$, 7—13, $\cdots$.

Any entry $gm + ht$ with $g \leqq 68$ has $h \leqq 25$ and is $< B$; any with $g \geqq 74$ has $h \geqq -2$ and is $> C$. Also, $70m + 15t$, $71m + 8t$, and $72m + 3t$ are $\leqq B$; $69m + 23t$, $70m + 17t$, $72m + 6t$, and $73m + 2t$ are $\geqq C$. This proves the theorem if $2m > 11t$.

Let $m = 2t + v$. In the case $6v < t < 8v$ in which there is a gap $> t$

in T_2 (and T_3), table T_4 contains no entry, not in T_2, which is between α and β of § 21, whence there is a gap $> t$ in T_4.

It remains only to consider the case $10t < 2m \leqq 11t$. Is $e_4 = t-1$? We obtain $d = 16$ by using only odd values β, $\beta+2$ $\cdots$, $\beta+14$ of b. Write $[j]$ for the set $j, j+1, \cdots, j+t-1$. Take $b = \beta, \beta+6, \beta+12, \beta+2, \beta+8, \beta+14, \beta+4, \beta+10$. We obtain the overlapping sets

$$[0]\ [6t-m]\ [12t-2m]\ [2t]\ [8t-m]\ [14t-2m]\ [4t]\ [10t-m].$$

The final number of the last set is $11t-m-1 \geqq m-1$. The same $d = 16$ is obtained by the method of § 21 and also by the use of modulus $8m$ instead of $2m$. We get

$$F = m^2\omega^2 - 408t^3m + 472t^2m^2 - 65142tm^3 - 105840m^4 + 24t^2m + 504tm^2 + 2646m^3,$$

where $\omega = D - 112t - 300m + 4$. Then $F > 0$ if $\omega \geqq 1887t$. But $A = D + 4C$, $C = 10m - 4t$, $\lambda = 4C + 112t + 300m = 340m + 96t$. Hence $F > 0$ if

$$A \geqq 1887t + \lambda + 381(m - 5t) = 721m + 78t.$$

We may lower this limit by subdivision of the interval from $5t$ to $5.5t$ for m. Making five equal divisions we find that the successive values of h in $\omega \geqq ht$ are 1757, 1789, 1822, 1855, 1887. To $h + \lambda$ add $356(m - lt) \geqq 0$ for $l = 5$, 5.1, 5.2, 5.3, 5.4, respectively. The respective limits for A are $696m + pt$, where $q = 73, 70, 67, 65, 61$. The essential point is the decrease, so that the maximum A is to be chosen.

Let the number of equal divisions increase indefinitely. Hence at the first step we have m approximately $5t$, and $F > 0$ if $\omega > 1724t$. We add $\lambda + 349(m - 5t)$ and obtain the best limit * $689m + 75t$ for A.

We test values of A below this limit by a table t_4 of certain sums by four of values of $f(x-4)$, which includes table T_2 to $144m$, table T_0 for larger numbers, the above entries $69m + \cdots$, and the following, in which $p = m - t$, $q = 6m - 3t$, $r = 10m - 4t$, and (x, y) denotes $xm + yt$:

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* Approximately 4 times the value $171m + 19t$ of $f(x)$.

$(10,0) = p+q+1+(3,3)$, $(16,4) = q+1+1+(10,5)$,
$(20,4) = p+q+(3,3)+(10,5)$, $(27,7) = r+(1,2)+(6,4)+(10,5)$,
$(30,8) = q+1+(3,3)+(21,7)$, $(35,4) = p+q+(28,8)$,
$(40,5) = p+r+(1,2)+(28,8)$, $(52,12) = q+1+(10,5)+(36,9)$,
$(76,14) = q+(1,2)+(3,3)+(66,12)$, $(104,20) = q+2(10,5)+(78,13)$,
$(148,28) = p+2(21,7)+(105,15)$, $(159,27) = p+(1,2)+(66,12)+(91,14)$,
$(164,28) = p+(3,3)+(55,11)+(105,15)$,
$(164,29) = p+(6,4)+(66,12)+(91,14)$,
$(168,28) = p+(10,5)+(21,7)+(136,17)$,
$(176,28) = p+(3,3)+(36,9)+(136,17)$,
$(191,27) = r+(6,4)+(55,11)+(120,16)$,
$(194,33) = p+(28,8)+(45,10)+(120,16)$,
$(200,31) = p+(3,3)+(91,14)+(105,15)$,
$(202,25) = 3m-2t+(28,8)+(171,19)$,
$(202,26) = p+(1,2)+(10,5)+(190,20)$,
$(202,27) = q+(10,5)+(15,6)+((171,19)$,
$(207,27) = p+(1,2)+(15,6)+(190,20)$,
$(228,29) = 3m-2+(105,15)+(120,16)$,
$(248,36) = p+(21,7)+(55,11)+(171,19)$,
$(260,36) = p+(10,5)+(78,13)+(171,19)$,
$(278,33) = p+(1,2)+(45,10)+(231,22)$,

where each (a, b) in the right members is a value of $f(x)$. In this t_4 the gaps are $\leqq t$ to $354m + 55t$. Write $m = 5t + v$. Then $2v \leqq t$. We employ the gaps $2v$ beginning with $(17, 13)$ [and ending with $(19, 3)$], $(33, 13)$, $(38, 15)$, $(53, 21)$, $(74, 23)$, $(90, 23)$, $(95, 25)$, $(106, 23)$, $(121, 29)$, $(136, 35)$, $(139, 33)$, $(142, 31)$, $(157, 37)$, $(166, 38)$, $(172, 39)$, $(173, 37)$, $(189, 37)$, $(194, 39)$, $(198, 41)$, $(200, 35)$, $(205, 37)$, $(210, 39)$, $(216, 43)$, $(226, 39)$, $(241, 45)$, $(257, 45)$, $(262, 47)$, $(273, 45)$, $(276, 43)$, $(278, 47)$, $(289, 45)$, $(294, 47)$, $(304, 51)$, $(310, 47)$, $(325, 53)$. Finally, we replace $(19, 4)$ by $18m + 9t + v$, and similarly replace $(27, 9)$, $(35, 4)$, $(97, 16)$, $(108, 15)$, $(108, 22)$, $(115, 25)$, $(123, 25)$, $(130, 27)$, $(169, 29)$, $(174, 31)$, $207, 33)$, $(212, 30)$, $(228, 31)$, $(237, 37)$, $(283, 41)$, $(298, 43)$, $(321, 45)$. Since there are no annexations * after $(278, 33)$, and a rarer use of the two devices, with none after $(325, 53)$, it is highly probable that the gaps are $\leqq t$ to the above limit $(689, 75)$ beyond which the gaps were proved $\leqq t$.

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* None are necessary after $(1542, 156)$. Taking $\delta = 8$ in (32), we find that $F > 0$ if $\omega \geqq 4094t$. But $\omega = A - 112t - 732m + 4$. We add $810(m - 5t)$ to the limit for A.

24. *Case* $k = 5$. First, let $m = 2t + v$, $6v < t < 8v$. Then T_5 contains

$$G = 121m + 13t = 15m - 5t + t + m + 2t + 105m + 15t.$$

We may now pass from α to β by gaps $\leqq t$ since $G - \alpha = 9v - t$ and $\beta - G = 2t - 8v$ are both > 0 and $< t$. By § 21, the gaps are $\leqq t$ to $400m + 58t$. For $e_4 = t - 1$, either of our methods gives $d = 18$.

Next, let $m = 5t + v$, $2v > t$. From T_5 we need

$54m$—5, —3, —2, 0, 1—15, 17—19, 21, 22; $55m$—19, —10, —8, —3, 0, 1, 3—9, 11—17, 19—23; $56m$—6, —3, 1—3, 5, 6, 8—23; $57m$—6, 0, 1—5, 7—13, 15—21, 23; $58m$—4, —2, —1, 0, 3—11, 13—23; $59m$—4, —3, —1, 0, 3—9, 11—17, 19—23; $60m$—20, —9, —1, 2, 4—7, 9—24;

while all earlier entries are $\leqq \alpha = 53m + 21t$, and in every later entry $gm + h$, $g \geqq 61$, $h \geqq -8$, whence it exceeds $\gamma = 56m + 5t$. Now $54m + 15t$, $55m + 9t$, $56m + 3t$, $57m - 6t$ are $\leqq \alpha$, $54m + 17t$, $55m + 11t$, $57m$, $58m - 4t$, $60m - 9t$ are $\geqq \gamma$. Hence the only possible entry between α and γ is $\delta = 60m - 20t$.

If $7v \leqq 6t$, we drop $\delta \leqq \alpha$ and have a gap $3m - 16t = 3v - t > 0$ from α to γ. If $7v > 6t$, $0 < \delta - \alpha = 7v - 6t \leqq t$, but $\gamma - \delta = 25t - 4m = 5t - 4v \geqq t$ since $t \geqq v$. Hence between α and γ there occurs some gap $> t$ if and only if

$$(52) \qquad 7v \leqq 6t < 9v \quad \text{or} \quad 7v > 6t,\ t > v.$$

But we can pass from α to γ by gaps $\leqq t$ if either $3v \leqq 2t$ or $v = t$.

If $3v \leqq 2t < 4v$, the gaps are $\leqq t$ at least to $354m + 55t$. The proof is similar to, but longer than, that at the end of § 23. For example, to $(304, 51)$ we add $3v - t \leqq t$ and obtain $(307, 35)$. Likewise from $(325, 53)$, we get $(328, 37)$, which is the sum of $m - t$, $(6, 4)$, $(45, 10)$, $(276, 24)$. We annex the sum $(312, 36)$ of $m - t$, $(3, 3)$, $(55, 11)$, $(253, 23)$. After these changes, the proof in § 23 shows that the gaps are $\leqq t$ from $(296, 50)$ to $(354, 55)$. The least v is 2 and then $t = 3$, $m = 17$, and either method gives $d = 22$ and hence a very large limit for A.

If $v = t$, then $e_4 = 2t - 1$. Since $m = 6t$, the maximum gap is the product of t by the maximum gap 2 when $t = 1$, the first one being then from 979 to 981 (*Bulletin*, II). This fact indicates the delicacy of the case $k \geqq 5$.

THEOREM 21. *If* $k = 5$, $t > 1$, $m \geqq 2t$, *the minimum of* $s + e_s$ *for* $s \geqq 4$ *is* L *of Theorem* 13 *when* $m \geqq 6t$ *and when* $m = 5t + v$, *subject to conditions* (52). *In all remaining cases, it is doubtless* $4 + e_4 = t + 3 = L - 1$, *but complete proof has not been given in special cases.*

25. THEOREM 22. *If $k>0$, $m\geqq 2t$, $t>1$, $3+e_3$ is equal to or greater than the minimum $s+e_s$ for $s\geqq 4$, except possibly when both $k\geqq 5$ and $m\leqq 4t$, a case not examined.*

We employ the existence of a gap $A=m-4t$ from $3t$ to $m-t$ if $m\geqq 4t$, gap $B=m-3t$ from $m+4t$ to $2m+t$ if $4t\leqq m\leqq 5t$, gap $C=2t$ from $2m-t$ to $2m+t$ if $m\geqq 5t$, gap $D=3t$ from $3m$ to $3m+3t$ if $m\geqq 6t$ or if $k=1$, $m\geqq 5t$, gap $E=3t$ from $5m+4t$ to $5m+7t$ if $k=1$, $m\geqq 3t$.

(I) $t>1$. First, let $m\geqq 7t$. By A, $g=3+E_3(m-t-1)=m-4t+2$. We use the notations of Theorem 13. If $u\geqq 7$, $v=0$, then $g-L=(u-5)(t-1)>0$. If $u\geqq 7$, $v\geqq 1$, $g-L=(u-5)(t-1)+v-1>0$.

Henceforth let $m=ut+v$, $2\leqq u\leqq 6$, $0\leqq v<t$. Theorems 17-21 state that the minimum M of $s+e_s$ for $s\geqq 4$ is either $t+4$ or $t+3$. Then $3+e_3\geqq M$ if $e_3>t$ or $>t-1$ and hence if there is a gap $>t+1$ or $>t$, respectively. If $u=5$ or 6, $C>t+1$.

Let $k=1$. If $u>2$, $E>t+1$. Let $u=2$. There is a gap $b=4t-m$ from $3m$ to $2m+4t$ if $m\leqq 4t$. Either $b=2t-v>t+1$ or $t-v=1$. In the latter case, $2v\geqq t$ and $M=t+3$ by Theorem 17, whence $b>t$.

Henceforth let $k>1$, $u\leqq 4$. First, let $u=4$. Then $B>t+1$ unless $m=4t$ or $4t+1$ and then $M=t+3$ by Theorems 18-21, whence $B>t$ if $m=4t+1$. When $m=4t$, there is a gap $2t$ beginning with $41t$ if $k=2$, $79t$ if $k=3$ or 4, $98t$ if $k=5$, $215t$ if $k=6$, $292t$ if $k=7$, $409t$ if $k=8$ or 9, etc.

There remain the cases $u=2$, $u=3$, with $k\geqq 2$. By Theorems 18-21, we have $M=t+3$ and seek a gap $>t$, except that $M=t+4$ if $u=2$, $k=2$, 3, 4, $m=2t+v$, $6v<t<8v$, and then we seek a gap $>t+1$.

Let $u=3$. If $2v\geqq t$, there is a gap $>t$ from $4m-2t$ to $3m+3t$. Hence let $2v<t$. If $k=2$ or 3, there is a gap $t+v$ from $\alpha=16m+6t$ to $\beta=17m+4t$, while if $v=0$ there is a gap $2t$ from $8m+8t=10m+2t=13m-7t$ to $10m+4t=12m-2t$. If $k\geqq 2$, there is a gap $3v$ from $2m+5t$ to $5m-4t$. Hence let $3v\leqq t$. Let also $k=4$. The only possible entries between α and β are $\gamma=20m-7t<\beta$ and $\delta=21m-9t\geqq\alpha$. First, let $t<4v$. Then $\delta>\beta$, $\gamma>\alpha$. While $\gamma-\alpha<t$, we have a gap $\beta-\gamma=2t-3v>t$ unless $t=3v$, and then there is a gap $4v>t$ from $22m+10t=250v$ to $23m+8t=26m-2t=254v$. Finally, if $k=4$, $t\geqq 4v$, there is a gap $t+v$ from $19m+12t$ to $20m+10t$. If also $v=0$, whence $m=3t$, there is a gap $2t$ from $112t$ to $114t$.

Let $u=2$. If $k=2$, there is a gap $4t-m$ from $7m$ to $6m+4t$. It

exceeds t and also $t+1$ if $t > 6v \neq 0$. For $k \geqq 2$, there is a gap $8t-3m$ from $5m-4t$ to $2m+4t$. It exceeds t if $t > 3v$ and exceeds $t+1$ if $t > 6v \neq 0$. Henceforth let $t \leqq 3v$.

Let $u=2$, $k=3$. The only possible entry between $\rho = 12m+3t$ and $\sigma = 10m+9t$ is $\tau = 15m-3t$. If $5v \geqq 2t$, then $\tau \geqq \sigma$ and the gap from ρ to σ is $2t-2v$, which exceeds t when $t > 2v$. But if $5v < 2t$, the gap from ρ to τ is $3v \geqq t$. When $3v = t$, then $m = 7v$ and there is a gap $4v > t$ from $14m+2t$ to $12m+8t$. Finally, let $t \leqq 2v$. The only possible entry between $\alpha = 16m+11t$ and $\beta = 17m+10t$ is $\gamma = 19m+3t$. If $3v \leqq 2t$, then $\gamma \leqq \alpha$ and there is a gap $\beta - \alpha = t+v > t$. But if $3v > 2t$, the gap from γ to β is $3t-2v > t$.

Let $u=2$, $k=4$. If $2t > 3v$, there is a gap $3t-3v > t$ from $22m+2t$ to $19m+11t$. Hence let $2t \leqq 3v$. If also $4v \leqq 3t$, there is a gap $3v-t \geqq t$ from $23m+11t = \delta$ to $26m+4t = \epsilon$. But if $4v > 3t$, $27m$ lies between δ and ϵ, while $\epsilon - 27m = 2t-v > t$. There remains only the case $3v = 2t$. There is always a gap $2v$ from $40m+14t$ to $42m+10t$. Here $2v > t$.

26. THEOREM 23. *If $t=1$, $k>0$, $m \geqq 2$, $3+e_3$ is equal to or greater than the minimum $s+e_s$ for $s \geqq 4$ except possibly when $m=4$, $k>9$, and when $m=3$, $k>14$.*

We employ gaps noted at the beginning of § 25. If $m \geqq 7$, $3+A-1 = m-2$ is the minimum M in Theorem 14. If $m=6$, $3+D-1=5=M$. The same holds also if $m=5$, $k=1$. But if $m=5$, $k \geqq 2$, $3+C-1 = 4 = M$. If $m=3$ or 4, $k=1$, $3+E-1=5 \geqq M$. If $m=3$ or 4, $k \geqq 2$, it suffices to exhibit a gap 2, whence $3+e_3 \geqq 4 = M$. For $m=4$, this was done up to $k=9$ in § 25, case $u=4$. For $m=3$, there is a gap 2 beginning with 32 if $k=2$ or 3, 112 if $k=4$, 134 if $k=5$ or 6, 449 if $k=7$ or 8, 1562 if $9 \leqq k \leqq 13$.

27. THEOREM 24. *If* $k=1, 2, 3, 4$, $m \geqq 2t$, $2+e_2$ is equal to or greater than the minimum M of $s+e_s$ for $s \geqq 4$.*

Consider a table R_k of sums by two of values of $f(x-k)$. In both R_1 and R_2 there is a gap from $\alpha = 39m+12t$ to $\beta = 42m+13t$ and $2+E_2(\beta-1) = 3m+t+1 \geqq M$ by Theorems 13, 14, 17, 18. In both R_3 and R_4 there is a gap from α to $\gamma = 42m+6t$, and $g = 2+E_2(\gamma-1) = 3m-6t+1$. If $m \geqq 3t$, then $g \geqq M$. If $t=1$, $m=2$, then $g=1$; but there is a gap from 5 to 8 in a table of sums of two squares, whence

* Also if $k=5$, except possibly when $t>1$, $m<3t$.

$2 + E_2(7) = 4 = M$. In R_5, the new entry $43m + 3t$ is $\geqq \gamma$ if $m \geqq 3t$ and then $g \geqq M$.

For $k = 1, 2, 3, 4$, and $m \leqq 6t$, there is a gap from $115m + 20t$ to $\delta = 120m + 16t$ and $2 + E_2(\delta - 1) = 5m - 4t + 1 \geqq t + 4$.

28. *Preliminary on $k > 0$, $m < 2t$.* The first remark in § 10 yields

THEOREM 25. *If $k > 0$, $m = t$, $s \geqq 3$, then $e_s = t - 1$.*

Since $k > 0$, we have $m \geqq t$ by § 2. Hence it remains to treat only the case $m > t$. It will greatly facilitate the discussion to write $m = t + q$ and employ the parameters t and q instead of t and m. We have $0 < q < t$. While the values $\geqq 2$ of k may be treated together (in Part IV), a separate discussion is necessary for $k = 1$.

PART III. THE CASE $k = 1$, $m = t + q$, $0 < q < t$, §§ 29-32.

29. THEOREM 26. *If $t \leqq 3q$, $e_5 = q - 1$. If $t \leqq (s-1)q$, $s \geqq 6$, $e_s = q - 1$.*

No smaller e_s will serve since (35) implies that no value of $f(x-1)$ lies between 0 and $q = f(-1)$.

We can find a small value of R in (37) which is congruent modulo m to any assigned integer.

If $s = 5$, we employ the values $t \equiv -q$, $m + 2t \equiv -2q \pmod{m}$, 0, q of $f(x-1)$. Adding $0, 1, \cdots, q-1$ to each, we get $4q \geqq m$ integers R congruent modulo m to the consecutive integers from $-2q$ to $2q - 1$ inclusive. The maximum such R is $m + 2t + q - 1$.

If $s \geqq 6$, we add $0, 1, \cdots, q-1$ to t, $2t$, $t + m + 2t$, 0, $q, \cdots, (s-4)q$ and get $sq \geqq m$ integers R congruent modulo m to the consecutive integers from $-3q$ to $(s-3)q - 1$. The maximum such R is the greater of $m + 3t + q - 1$ and $(s-3)q - 1$.

We choose R so that $A = N - R$ is congruent to $4C + tb$, whence (44) determines an integer $(a - 3b)/2$. Since $n < 0$, F in (24) is positive for every D. Take $A \geqq 4m - 2t$. Then (43) holds since $(m-t)^2 + 3t^2 > 0$. It remains only to examine integers less than the sum of $4m - 2t$ and the preceding maximum R. If $s \geqq 5$, they are less than the greater of $6t + 6q$—1 and $2t + (s+1)q - 1$.

We obtain overlapping sets of consecutive integers if we add $0, 1, \cdots,$

$q-1$ to the following seven groups of sums of s of $0, q, t, \kappa = m+2t = 3t+q$:

$$jt, jt+q, \cdots, jt+(s-2)q \qquad (j=0,1,2);$$
$$3t, \kappa, \kappa+q, \cdots, \kappa+(s-1)q;$$
$$\kappa+t, \kappa+t+q, \cdots, \kappa+t+(s-2)q;$$
$$\kappa+2t, \kappa+2t+q, \cdots, \kappa+2t+(s-3)q;$$
$$6t \text{ (if } s>5), \kappa+3t, 2\kappa, 2\kappa+q, 2\kappa+2q, 2\kappa+3q.$$

30. THEOREM 27. *If $k=1$, $m=t+q$, $0<q<t$, then $e_5=t-2q-1$ for $t\geqq 3q$, $e_s=t-(s-2)q-1$ for $t\geqq(s-1)q$, $s\geqq 6$.*

No smaller value of e_s will serve. For, if $s\geqq 6$, no sum of s values of $f(x-1)$ lies between $\rho=2t+(s-2)q$ and $3t$. In fact, $0, q, t$ are the only values of f less than $3t$. But if we use a single t, the maximum sum is $t+(s-1)q<\rho$, and $sq<\rho$. Next, if $s=5$, no sum of s values lies between $5t+3q$ and $6t+q$. The values not too large are $0, q, t, \kappa=3t+q$. The sum 2κ is too large. The case of a single summand κ is excluded, since no sum by four of $0, q, t$ lies between $2t+2q$ and $3t$. Finally, if κ is not used, the maximum sum $5t$ is too small.

We can find a small value of R in (37) which is congruent modulo m to any assigned integer.

If $s=5$, we add $0, 1, \cdots, e_5$ to q, and add only $0, 1, \cdots, q-1\leqq e_5$ to 0, $t\equiv -q$, $m+2t\equiv -2q \pmod m$. We get $m=t+q$ integers R congruent modulo m to the consecutive integers from $-2q$ to $t-q-1$ inclusive. The maximum such R is $m+2t+q-1$.

If $s\geqq 6$, we add $0, 1, \cdots, e_s$ to $(s-4)q$, but add only $0, 1, \cdots, q-1\leqq e_s$ to $t, 2t, t+m+2t, 0, q, \cdots, (s-5)q$. We get $m=t+q$ integers R congruent modulo m to the consecutive integers from $-3q$ to $t-2q-1$ inclusive. The maximum such R is $\rho=m+3t+q-1$.

We proceed as in § 29. It remains to examine integers $<4m-2t+\rho=\sigma$. We employ the seven groups of sums of s values of $f(x-1)$ given at the end of § 29. To the final entry in each group except the last, we add $0, 1, \cdots, e_s$; to all remaining entries we add only $0, 1, \cdots, q-1$. We obtain seven overlapping sets of consecutive integers which include $0, 1, \cdots, \sigma=6t+6q-1$.

31. THEOREM 28. *For $k=1$, $m=t+q$, $0<q<t$, write L for the least $s+e_s$ when $s\geqq 5$. When $q=1$, $L=t+1$ if $t\geqq 5$ (given by $s=6, 7, \cdots$, or $t+1$), $L=t+2$ if $t=3$ or 4 $(s=5)$, $L=5$ if $t=2$ $(s=5)$. When $q>1$, write $t=pq+\rho$, $0\leqq\rho<q$. Then $L=q+q+1$ if $p=4$ or if $p\geqq 5$, $\rho>0$ $(s=p+2)$; $L=p+q$ if $p\geqq 5$, $\rho=0$ $(s=p+1)$;*

$L = q + 5$ *if* $p = 3$, $\rho > 0$ $(s = 6)$; $L = q + 4$ *if* $p = 3$, $\rho = 0$, *or if* $p = 1$ or 2 $(s = 5)$.

The proof by means of Theorems 26 and 27 is similar to that in §§ 7, 15. We next seek the cases in which $4 + e_4 < L$.

THEOREM 29. *With the notations of Theorem* 28, *we have* $4 + e_4 \geqq L$ *if* $t > 2q$, *but* $e_4 = q - 1$, $4 + e_4 = L - 1$, *if* $2t \leqq 4q \leqq 3t$ *and doubtless if merely* $t \leqq 2q$, *with certainty for integers* $\leqq 473t + 427q$.

In a Table T_1 of sums by four of values of $f(x - 1)$, there is a gap from $5t + 2q$ to $6t + q$, and $\lambda \equiv 4 + E_4(6t + q - 1) = t - q + 3$.

First, let $t = 2q$, whence $m = 3q$, $f(x) = q \cdot e(x)$, where $e(x) = p_5(-x)$ is an extended pentagonal number. By *Bulletin*, III, every integer $\geqq 0$ is a sum of four numbers chosen from 1 and values of $e(x)$. Hence every multiple of q is a sum of four numbers chosen from $q = f(-1)$ and values of $f(x)$. Hence $e_4 = q - 1$.

Second, let $t > 2q$. If $q = 1$, $\lambda = t + 2 \geqq L$. Next, let $q > 1$, whence $p \geqq 2$ in Theorem 28. Then

$$\lambda - (p + q + 1) = (p - 2)(q - 1) + \rho \geqq 0.$$

Hence if $\rho = 0$, $\lambda - (p + q) > 0$. If $p = 3$, $\rho > 0$, $\lambda - L = \rho + q - 2 > 0$. If $p = 3$, $\rho = 0$, $\lambda - L = q - 1 > 0$. Finally, if $p = 2$, then $\rho > 0$, $\lambda - L = \rho - 1 \geqq 0$. In all these cases, $\lambda \geqq L$.

Third, let $t < 2q$. Then $t = q + \rho$, $0 < \rho < q$. By (39), $n = -4q - \rho$, $C = q$. Then

$$U = 24mD - 48q\rho - 15\rho^2,$$
$$V = 2mD + 12q^2 + 6q\rho + \rho^2 + 4q + 2\rho.$$

(I) Let $q \leqq 2\rho$. We shall prove that $e_4 = q - 1$. Write $[j]$ for the set $j, j + 1, \cdots, j + q - 1$. For $b = \beta, \beta + 2, \beta + 4, \beta + 6$, and $r = 0, 1, \cdots, q - 1$, the values of $bt + r$ are congruent modulo m to βt plus the numbers in the overlapping sets $[0]$, $[\rho]$, $[2\rho]$, $[3\rho]$, since $2t \equiv \rho$. The largest number of the last set is $3\rho + q - 1 \geqq m - 1$. Since the four sets include a complete set of residues modulo m, we have $d = 8$. Then

$$P = 24q + 13\rho, \qquad W = 3mD - 72q^2 - 84q\rho - 23\rho^2,$$

$$F = m^2D^2 - 960mq^2D - 1056mq\rho D - 288m\rho^2D + 16mqD + 8m\rho D$$
$$+ 2304q^4 + 7488q^3\rho + 7668q^2\rho^2 - 768q^3 - 1344q^2\rho + 3162q\rho^2$$
$$- 756q\rho^2 + 456\rho^4 - 138\rho^3 + 64q^2 + 64q\rho + 16\rho^2.$$

The terms involving D suggest the substitution

$$mD = 960q^2 + 1056q\rho + 288\rho^2 - 16q - 8\rho - p.$$

Multiply this by mD and make the same substitution in pmD. Write $p = xq^2 + yq\rho + z\rho^2$. In F the coefficients of the terms of degree 4 in q and ρ are now all positive and small if $x = 2$, $y = 5$, $z = 1$. Then

$$F = 169\rho^4 + 676q\rho^3 + 881q^2\rho^2 + 596q^3\rho + 388q^4 - 736q^3 - 1248q^2\rho - 700q\rho^2 - 130\rho^3.$$

If $\rho = 1$, then $q = 2$, $F > 0$. If $\rho \geqq 2$, the term involving $q^i\rho^j$ is numerically less than that involving $q^i\rho^{j+1}$ for $i, j = 0, 1, 2, 3$, whence $F > 0$. Division of mD by $m = 2q + \rho$ yields the remainder $-2q\rho$ and quotient $479q+287\rho-8$. Since $A = D + 4q$, $F > 0$ if

$$A \geqq 483q + 287\rho + 20(q - p) = 503q + 267\rho.$$

To this point in table T_1 the gaps were verified to be $\leqq q$.

(II) Let $3\rho \geqq q > 2\rho$. Is $e_4 = q - 1$? The use of the further value $b = \beta + 8$ and hence the set $[4\rho]$ gives $d = 10$ as in (I). Also the method of § 21 yields $d = 10$. We find that

$$F = m^2D^2 - 1744mq^2D - 1868mq\rho D - 498m\rho^2D + 16mqD + 8m\rho D + 10816q^4 + 27616q^3\rho + 24856q^2\rho^2 + 1311\rho^4 + 9498q\rho^3 - 1664q^3 - 2720q^2\rho - 1460q\rho^2 - 258\rho^3 + w^2, \quad w = 8q + 4\rho.$$

Proceeding as in (I), we get $x = 6$, $y = 9$, $z = 2$, and find that $F > 0$ if $D \geqq 869q + 496\rho$. But $A = D + 4q$. Thus $F > 0$ if

$$A \geqq 873q + 496\rho + 15(q - 2\rho) = 888q + 466\rho.$$

To this point (and on to $900q + 473\rho$), it was verified that all gaps in T_1 are $\leqq q$ when $\rho \leqq q$.

(III) Let $q > 3\rho$. Since the preceding verification becomes nearly automatic for large numbers, it is highly probable that $e_4 = q - 1$. Proof would be long since $d \geqq 12$ by either method.

32. THEOREM 30. *If* $k = 1$, $m = t + q$, $0 < q < t$, *both* $3 + e_3$ and $2 + e_2$ *are* $\geqq L$ *of Theorem* 28.

In a table of sums by three of values of $f(x-1)$, there is a gap from $5t + q$ to $\alpha = 6t + 2q$, and $3 + E_3(\alpha - 1) = t + q + 2 \geqq L$ in all cases. In a table of sums by two, there is a gap from $4t + q$ to α, and $2 + E_2(\alpha - 1) = 2t + q + 1$, which exceeds $t + q + 2 \geqq L$.

COROLLARY. *If $t > 2q$, the minimum of all $s + e_s$ for $s \geqq 1$ is L of Theorem 28 and is given by a value $\geqq 5$ of s. But if $t \leqq 2q$, the minimum is $4 + e_4 = q + 3$.*

PART IV. THE CASE $k \geqq 2$, $m = t + q$, $0 < q < t$, §§ 33-41.

33. THEOREM 31. *If $s \geqq 5$, $t \leqq (s+1)q$, then $e_s = q - 1$.*

No smaller e_s will serve since no value of $f(x-k)$ lies between 0 and q. For, if $p > 2$, $f(-p)$ exceeds $\epsilon = f(-2) = 3m - 2t = t + 3q$.

There exists a small value of R in (37) which is congruent modulo m to any assigned integer.

If $s = 5$, we employ the following values of $f(x-k)$:

$$(53) \qquad \epsilon \equiv 2q,\ q,\ 0,\ t \equiv -q,\ m + 2t \equiv -2q,\ 3m + 3t \equiv -3q,\ 6m + 4t \equiv -4q \pmod m.$$

Adding $0, 1, \cdots, q-1$ to each, we get $7q \geqq m$ integers R which are congruent modulo m to the consecutive integers from $-4q$ to $3q-1$ inclusive. The maximum such R is $6m + 4t + q - 1$.

If $s \geqq 6$, we employ

$$(54) \qquad 2\epsilon + (s-6)q \equiv (s-2)q,\ \epsilon + (s-5)q \equiv (s-3)q,\ (s-4)q, \cdots, q,\ 0,\ t \equiv -q,\ 2t \equiv -2q,\ t + (m+2t) \equiv -3q \pmod m.$$

Adding $0, 1, \cdots, q-1$ to each, we get $(s+2)q \geqq m$ integers R which are congruent modulo m to the consecutive integers from $-3q$ to $(s-1)q - 1$ inclusive. The maximum such R is the greater of $2t + (s+1)q - 1$ and $m + 3t + q - 1$.

First, let $k = 2$. Then

$$(55) \qquad n = 2t - 5m,\quad C = 3m - 2t,\quad A = D + 4C.$$

We employ a single odd b and apply § 4 with $d = 2$, $e_4 = 0$. Then P is negative, and only the minor conditions are to be satisfied. We have

$$16V - U = 8mA + 151m^2 - 100mt + 28t^2 - 32me_4 > 0,$$

for every $A \geqq 0$. Here (14) becomes

$$3A \geqq 23m - 8t - 4t^2/m,$$

which holds if $A \geqq 8m - 3t$. Then

$$U = 24mA + 36t^2 + 36mt - 135m^2 > 0.$$

It remains only to examine the integers less than the sum of $8m-3t$ and the preceding maximum R. For $s \geqq 5$, these are less than the greater of $15t+15q-1$ and $7t+(s+9)q-1$. The left column of the following table contains only sums of s values of $f(x-2)$. Consecutive entries in any line differ by q. To each entry we add $0, 1, \cdots, q-1$. We therefore obtain a set of consecutive integers terminating with $M-1$, where M is listed in the right column in a relation $M \geqq Q$ found by use of $(s+1)q \geqq t$. Since Q is the first entry of the next line of the first column, the sets of consecutive integers overlap, and hence together include all integers from 0 to $15t+15q-1$.

$0, q, \cdots, sq$	$(s+1)q \geqq t$
$t,\ t+q,\ t+2q,\ \epsilon+jg \quad (j=0,\cdots,s-3)$	$t+(s+1)q \geqq 2t$
$2t, 2t+q, 2t+2q, t+\epsilon+jg$	$2t+(s+1)q \geqq 3t$
$3t, 3t+q, 3t+2q, 2t+\epsilon+jq$	$3t+(s+1)q \geqq 4t$
$4t, \kappa+t+lq \ (l=0,1,2), \epsilon+\kappa+jq$	$4t+(s+2)q \geqq 5t+q$
$\kappa+2t+lq, t+\epsilon+\kappa+jq$	$5t+(s+2)q \geqq 6t+q$
$\kappa+3t, 2\kappa, \lambda+pq \ (p=0,\cdots,s-1)$	$6t+(s+3)q \geqq 7t+2q$
$2\kappa+t, \lambda+t+lq, \epsilon+\lambda+uq \ (u=0,\cdots,s-2)$	$7t+(s+5)q \geqq 8t+4q$
$\lambda+2t+q, \lambda+2t+2q, \epsilon+\lambda+t+jq$	$8t+(s+4)q \geqq 9t+3q$
$3\kappa+lq, \epsilon+\lambda+2t, \epsilon+\lambda+2t+q,$ $2\epsilon+2\kappa+t, 2\epsilon+\lambda+t+wq \ (w=0,\cdots,s-4)$	$9t+(s+6)q \geqq 10t+5q$
$\kappa+\lambda+t+q, \mu+pq$	$10t+(s+6)q \geqq 11t+5q$
$\kappa+\lambda+2t+q, \mu+t+uq$	$11t+(s+5)q \geqq 12t+4q$
$4\kappa, 4\kappa+q, 2\lambda+uq$	$12t+(s+5)q \geqq 13t+4q$
$4\kappa+t, 2\kappa+\lambda+t, 2\kappa+\lambda+t+q, \kappa+\mu+uq$	$13t+(s+6)q \geqq 14t+5q$
$2\kappa+\lambda+2t, 2\lambda+2t+zg \ (z=0,1),$ $\kappa+\mu+t+q+zq, \kappa+\epsilon+\mu+jq$	$14t+(s+8)q \geqq 15t+7q$
$\kappa+2\lambda+lq, \nu+pq$	$15t+(s+10)q \geqq 16t+9q$

The case $k>2$ of Theorem 31 is a corollary to the case $k=2$.

34. THEOREM 32. *If* $k \geqq 2$, $m=t+q$, $0<q<t$, $t \geqq (s+1)q$, $s \geqq 5$, *then* $e_s = t-sq-1$.

No smaller value of e_s will serve. For, 0 and q are the only values $<t$ of $f(x-k)$, whence no sum of s values lies between sq and t.

If $s=5$, add $0, 1, \cdots, e_5$ to the first number ϵ in (53), but add only $0, 1, \cdots, q-1 \leqq e_5$ to the others. We get $t+q=m$ integers R which are congruent modulo m to the consecutive integers from $-4q$ to $t-3q-1$ inclusive. The maximum such R is $6m+4t+q-1$.

If $s \geqq 6$, we proceed similarly with (54) instead of (53), and get $t+q=m$ integers R which are congruent modulo m to the consecutive integers from $-3q$ to $-2q-1$ inclusive. The maximum such R is $m+3t+q-1$.

As in § 33, it remains only to examine the integers less than the sum $15t+15q-1$ of $8m-3t$ and the larger of the preceding maxima. We employ the left column of the table in § 33. We now add $0, 1, \cdots, e_8$ to the last entry of each line and obtain $Q-1$, where Q is the second member of the inequality in the right column.

35. THEOREM 33. *When $k \geqq 2, m = t+q, 0 < q < t$, write L for the least $s+e_s$ for $s \geqq 5$. When $q=1$, $L = t-1$ if $t \geqq 6$ (given by $s=5, 6, \cdots$, or $t-1$); $L=5$ if $t=2$, 3, 4, or 5 ($s=5$). When $q>1$, write $t = pq+\rho$, $0 \leqq \rho < q$. Then $L = p+q-1$ if $p \geqq 6$, $\rho \geqq 1$ ($s=p$); $L = p+q-2$ if $p \geqq 6$, $\rho = 0$ ($s=p-1$); $L = q+4$ if $1 \leqq p \leqq 5$ ($s=5$).*

This follows from Theorems 31 and 32.

36. THEOREM 34. *If $q=1$, $4+e_4 < L$ only when $k \geqq 2$, $t=2$ or 3, and when $k \geqq 3$, $t=4$, and then $4+e_4 = 4 = L-1$.*

Here $t = m-1$, $f(x) = e(x) = \frac{1}{2}m(x^2+x)-x$ is an extended polygonal number. By *Bulletin*, III, when $k=2$, $e_4 = t-5$ if $t \geqq 7$, $e_4 = 2$ if $t=6$, $e_4 = 1$ if $t=4$ or 5, $e_4 = 0$ if $t=2$ or 3; when $k \geqq 3$, $e_4 = t-5$ if $t \geqq 6$, $e_4 = 1$ if $t=5$, $e_4 = 0$ if $t=2$, 3, 4.

37. THEOREM 35. *When $m = t+q$, $1 < q < t$, then $4+e_4 \geqq L$ if $p \geqq 5$, $k \geqq 2$; and if $k=2$, $p=4$, or $p=3$, $\rho > 0$. But if $t = 3q$, $k \geqq 2$, $e_4 = q-1$.*

In a table T_k of sums by four of values of $f(x-k)$, there is a gap from $4q$ to t if $p \geqq 4$. First, let $p \geqq 6$. Then

$$\lambda = 4 + E_4(t-1) = pq+\rho-4q+3$$

exceeds L by $(p-5)(q-1)+\delta > 0$, where $\delta = \rho-1$ if $\rho \geqq 1$, but $\delta = 0$ if $\rho = 0$. Second, let $p=5$. Then $\lambda - L = \rho - 1$. But if $p=5$, $\rho = 0$, then $t = 5q$, $f(x) = q \cdot e(x)$, for $e(x)$ as in § 36 with $m=6$, whence its e_4 is 1. In other words, a table of sums by four of values of $e(x-k)$ has the maximum gap 2 if $k \geqq 2$. Hence T_k has the maximum gap $2q$. Thus, for $f(x-k)$, $e_4 = 2q-1$, $4+e_4 > L$.

In the following table T_2, the terms free of t are to be multiplied by q:

0—4; t+0—6; $2t$+0—8; $3t$+0—4, 6, 7, 9, 10; $4t$+0—6, 9, 12;
$5t$+1, 2, 4, 5, 7, 8; $6t$+1—7, 10; $7t$+2—8; $8t$+2—10; $9t$+3—6, 9, 12;
$10t$+3—9; $11t$+4, 6—11; $12t$+4—10, 12, 13; $13t$+5—10, 12, 15;
$14t$+6—12; $15t$+6—8, 10—13; $16t$+7—15.

All entries $xt+yq$ with $x \geqq 17$ have $y \geqq 8$ and exceed $z = 15t + 10q$ since $t > q$.

If $p = 4$, $k = 2$, there is a gap from $5t + 2q$ to $y = 5t + 4q$, and $4 + E_4(y-1) = 2q + 3 > L$. If $p = 3$, $k = 2$, there is a gap from $14t + 12q$ to z, and $4 + E_4(z-1) = q + \rho + 3 \geqq L$ if $\rho > 0$. But if $\rho = 0$, then $t = 3q$ and $f(x) = q \cdot e(x)$, with $m = 4$ in the $e(x)$ of § 36. Since every integer $\geqq 0$ is a sum of four values of $e(x-k)$, every multiple of q is a sum of four values of $f(x-k)$, whence $e_4 = q - 1$.

38. THEOREM 36. *Let* $m = t + q$, $t = pq + \rho > q$, $p = 1$ *or* 2, $0 \leqq \rho < q$, $k \geqq 2$. *Then* $e_4 = q - 1$.

No smaller e_4 will serve since there is a gap from 0 to q. If $p = 1$ or if $p = 2$, $\rho = 0$, then $t \leqq 2q$ and the theorem is a corollary to Theorem 29.

Let $p = 2$, $\rho > 0$, $k = 2$. We have (55). Thus

$$n = -11q - 3\rho, \quad U = 24mD + 585q^2 + 282q\rho + 33\rho^2,$$
$$C = 5q + \rho, \quad V = 2mD + 115q^2 + 64q\rho + 9\rho^2 + 6q + 2\rho.$$

We obtain the smaller d by the method of § 21. We employ only $b = \beta, \cdots, \beta + 7$ and hence suppress the final pair in (46) and the final set in (50) and (51). When $\beta \equiv 1 \pmod 4$, there are two cases. If $2\rho \leqq q$, the sets

$$[0],\ [6t-4m] = [2\rho],\ [2t-m] = [\rho+q],\ [t] = [\rho+2q],\ [m] = [\rho+3q],$$
$$[3t-m] = [2\rho+3q],\ [2t] = [2\rho+4q],\ [5t-2m] = [3\rho+4q],$$
$$[4t-m] = [3\rho+5q]$$

overlap and terminate with $3\rho + 6q - 1 > 2m - 1$. But if $2\rho > q$, we insert $[7t - 5m] = [2\rho - q]$ after $[0]$. If $\beta \equiv 3 \pmod 4$, the sets

$$[0],\ [3t-2m] = [\rho],\ [2t-m] = [\rho+q],\ [5t-3m] = [2\rho+q],$$
$$[4t-2m] = [2\rho+2q],\ [7t-4m] = [3\rho+2q],$$
$$[6t-3m] = [3\rho+3q],\ [2t] = [2\rho+4q],\ [t+m] = [2\rho+5q]$$

overlap and terminate with $2\rho + 6q = 2m - 1$. If $\beta \equiv 0 \pmod 4$, the set $[4t - m] = [3\rho + 5q]$ terminates with $3\rho + 6q - 1 \equiv \rho - 1 \pmod{2m}$ and includes $0, 1, \cdots, \rho - 1$. We employ these and

$$[3t-2m] = [\rho],\ [6t-4m] = [2\rho],\ [5t-3m] = [2\rho+q],$$
$$[t] = [\rho+2q],\ [m] = [\rho+3q],\ [3t-m] = [2\rho+3q],$$
$$[2t] = [2\rho+4q],\ [t+m] = [2\rho+5q],$$

which overlap and terminate with $2m-1$. If $\beta \equiv 2 \pmod 4$, the sets

$$[0],\ [3t-2m]=[\rho],\ [2t-m]=[\rho+q],\ [5t-3m]=[2\rho+q],$$
$$[4t-2m]=[2\rho+2q],\ [3t-m]=[2\rho+3q],$$
$$[6t-3m]=[3\rho+3q],\ [5t-2m]=[3\rho+4q],\ [t+m]=[2\rho+5q]$$

overlap and terminate with $2m-1$. Hence $d=8$ in every case. Then

$$P=31q+11\rho,\quad W=3mD-47q^2-50q\rho-11\rho^2,$$
$$F=m^2D^2-1368mq^2D-1008mq\rho D-184m\rho^2D+24mqD+8m\rho D$$
$$-33786q^4-41322q^3\rho-18462q^2\rho^2-3558q\rho^3-248\rho^4$$
$$+882q^3+474q^2\rho+30q\rho^2-10\rho^3+w^2,\quad w=12q+4\rho.$$

Proceeding as in (I) of § 31, we get $x=25$, $y=11$, $z=2$,

$$F=1039q^4-524q^3\rho+183q^2\rho^2+526q\rho^3+124\rho^4$$
$$+282q^3+10q^2\rho-106q\rho^2-26\rho^3+w^2>0.$$

Hence $F>0$ if $mD\geqq 1393q^2+1019q\rho+186\rho^2-24q-8\rho$, and hence if $D\geqq 465q+186\rho$. Adding $4C=20q+4\rho$, we get $F>0$ if

$$A\geqq 485q+190\rho+15(q-\rho)=500q+175\rho.$$

It was verified that all gaps are $\leqq q$ in T_2 to this point.

39. *Case* $p=3$. First, let $k=3$. We use the following part of table T_3 of sums by four of values of $f(x-3)$:

$48t$+31—45; $49t$+32—38, 40—43, 45, 46; $50t$+32—40, 42, 43, 45, 48;
$51t$+33—36, 38—44, 48, 49; $52t$+34—43, 46, 48, 51;

in which terms free of t are to be multiplied by q. All entries $gt+hq$ with $g<48$ have $h\leqq 43$ except $45t+46q$, and all such are $<\alpha=50t+40q$. To show that $\gamma=49t+45q$ is the only possible entry between α and $\delta=52t+34q$, note that $48t+45q<\alpha$, $49t+43q\leqq\alpha$, $49t+46q>\delta$, $50t+42q>\delta$, $51t+36q<\alpha$, $51t+38q>\delta$. All entries $gt+hq$ with $g\geqq 53$ have $h\geqq 36$ and exceed δ.

If $2q<3\rho$, then $\gamma<\delta$ and there is a gap $2q-\rho>q$ from α to γ. But if $2q\geqq 3\rho$, the gap is from α to δ and is 2ρ.

Hence there remains only the case $2\rho\leqq q$. We shall prove that $e_4=q-1$. The use of only odd values of b yields too large a value of d. We obtain the small value $d=9$ by using the same sets as in § 21, but expressed in terms of q and ρ.

$\beta \equiv 1 \pmod 4$: $[0]$, $a = [8t - 6m] = [2\rho]$, $b = [7t - 5m] = [2\rho + q]$, $c = [2t - m] = [\rho + 2q]$, $[t]$, $[m]$, $f = [8t - 5m] = [3\rho + 4q]$, $g = [3t - m] = [2\rho + 5q]$, $h = [2t] = [2\rho + 6q]$, $i = [6t - 3m] = [3\rho + 6q]$, $j = [5t - 2m] = [3\rho + 7q]$.

$\beta \equiv 3 \pmod 4$: $[0]$, a, $k = [3t - 2m] = [\rho + q]$, c, $l = [6t - 4m] = [2\rho + 2q]$, $p = [5t - 3m] = [2\rho + 3q]$, $r = [4t - 2m] = [2\rho + 4q]$, f, $s = [7t - 4m] = [3\rho + 5q]$, h, $u = [t + m] = [2\rho + 7q]$.

$\beta \equiv 0 \pmod 4$: j, $v = [4t - 3m] = [\rho]$, k, b, l, p, $[m]$, f, g, h, u.

$\beta \equiv 2 \pmod 4$: $[0]$, a, k, c, $[t]$, p, r, g, s, i, u.

In each case the sets overlap, terminate with a number $\geqq 2m - 1$, and hence include a complete set of residues modulo $2m$.

In view of its application below, we shall prove that $E_4(A) = q - 1$ when $2\rho > q$ for every A not between the above α and δ. If $\beta \equiv 1 \pmod 4$, insert the sets $v = [\rho]$ and r after $[0]$ and $[m]$ respectively. If $\beta \equiv 3 \pmod 4$, insert only v. If $\beta \equiv 0 \pmod 4$, insert only r. In these three cases, the use of nine consecutive values of b leads to a complete set of residues modulo $2m$. But this fails if $\beta \equiv 2 \pmod 4$, since the only available sets not used above are j and b, neither of which contains any one of the now missing numbers between $q - 1$ and 2ρ. All of the latter occur in the new set $[9t - 7m] = [2\rho - q]$ obtained from the new value $\beta + 9$ of b. Hence we may take $d = 10$ in all cases.*

By taking $d = 10$, we may treat together the above two cases $2\rho \leqq q$ and $2\rho > q$. We shall find a limit on A beyond which $e_4(A) = q - 1$. For $k = p = 3$, $e_4 = q - 1$, we have

$$n = -22q - 5\rho, \quad U = 24mD + 3108q^2 + 1380q\rho + 153\rho^2,$$
$$C = 15q + 3\rho, \quad V = 2mD + 476q^2 + 218q\rho + 25\rho^2 + 8q + 2\rho,$$
$$P = 50q + 13\rho, \quad W = 3mD + 76q^2 + 10q\rho - 2\rho^2,$$
$$F = m^2D^2 - 3248mq^2D - 1748mq\rho D - 234m\rho^2D + 32mqD + 8m\rho D$$
$$- 422624q^4 - 417448q^3\rho - 153764q^2\rho^2 - 25038q\rho^3 - 1521\rho^4$$
$$+ 8032q^3 + 5240q^2\rho + 1120q\rho^2 + 78\rho^3 + w^2, \quad w = 16q + 4\rho.$$

Write $mD = 3248q^2 + 1748q\rho + 234\rho^2 - 2w + p$. Multiply this by mD and make the same replacement in pmD. Take $p = xq^2 + yq\rho + z\rho^2$. In F the coefficients of the terms of degree 4 in q and ρ are small and positive if $x = 126$, $y = 57$, $z = 7$. Then

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* No improvement results by working modulo $8m$.

$$F = 2500q^4 + 2300q^3\rho + 3105q^2\rho^2 + 1334q\rho^3 + 166\rho^4 + 4000q^3 + 2408q^2\rho + 440q\rho^2 + 22\rho^3 + w^2.$$

Hence $F > 0$ if $mD \geqq 3374q^2 + 1805q\rho + 241\rho^2$. Division by $m = 4q + \rho$ yields the remainder $-2q^2 - 3q\rho$ and quotient $844q + 241\rho$. But $A = D + 4C$. Hence $F > 0$ if

$$A \geqq 904q + 253\rho + 15(q - \rho) = 919q + 238\rho.$$

To this entry of T_3 the gaps are $\leqq q$ except between α and δ. This proves the above statements.

For $k = 4$, we annex to T_3 the sum

$$50t + 41q = t + 6t + 10q + 15t + 10q + 28t + 21q$$

of four values of $f(x - 4)$. Now the gaps are $\leqq q$ between α and δ. Hence the above statements imply $e_4 = q - 1$.

THEOREM 37. *Let $m = t + q$, $t = 3q + \rho$, $0 < \rho < q$. If $k = 3$ and $2\rho > q$, then $4 + e_4 \geqq L$. If $k \geqq 3$ and $2\rho \leqq q$, then $e_4 = q - 1$, $4 + e_4 = L - 1$. The latter holds always if $k \geqq 4$.*

40. *Case $p = 4$.* First, let $k = 3$. From T_3 we need

$12t+4$—16, 19, 24; $13t+5$—10, 12—14, 15, 18; $14t+6$—13, 15, 16;
$15t+6$—8, 10—13, 15, 16, 18, 21; $\cdots$; $23t+12$—16, 18, 19, 21, 22;
$24t+12$—24, 28; $25t+13$, 14, 16—22, 24, 25; $26t+14$—17, 19—22, 24, 27.

and the following facts. The only entries $xt + yq$ with $x < 12$, $y > 18$ are $9t + 19q$ and $10t + 21q$, both $< \alpha = 13t + 18q$. All entries with $x > 15$ have $y \geqq 7$ and exceed $\beta = 15t + 10q$. All with $x \leqq 22$ have $y \leqq 25$ and are $< \delta = 25t + 22q$. All with $x \geqq 27$ have $y \geqq 15$ and are $\geqq \epsilon = 26t + 19q$. Hence the only possible entry between α and β is $\gamma = 12t + 24q$. If $2q < 3\rho$, then $\gamma < \beta$ and there is a gap $2q - \rho > q$ from α to γ. Hence let $2q \geqq 3\rho$. Then there is a gap 2ρ from α to β. Hence let $2\rho \leqq q$. Then there is a gap $q + \rho$ from δ to ϵ. But if $\rho = 0$, then $t = 4q$, $m = 5q$, and, as at the end of § 37 with $m = 5$ instead of $m = 4$, we have $e_4 = q - 1$.

Next, let $k = 4$. From T_4 we need

$68t+47$, 49—59, 61, 62, 65, 68; $69t+48$—50, 52—58, 60—65;
$70t+49$—53, 55—57, 59—62, 64—66, 71;

and the facts that every entry $xt + yq$ with $x \leqq 67$ has $y \leqq 66$ and is $\leqq \phi = 69t + 58q$, while every one with $x \geqq 71$ has $y \geqq 50$ and is $\geqq \psi =$

$71t + 50q$. Hence there is a gap * 2ρ from ϕ to ψ. But if $2\rho \leqq q$, it was verified that all gaps are $\leqq q$ to $200t + 185q$. Then an attempt to prove that $e_4 = q - 1$ leads to $d = 16$ by use of $b = \beta + j$, $j = 0, 10, 4, 14, 8, 2, 12, 6$. The method of § 21 gives the same d. But the limit for A would be very large.

Finally, if $k = 5$ and $2\rho > q$, there is a gap 2ρ from $85t + 74q$ to $87t + 66q$.

THEOREM 38. *Let* $m = t + q$, $t = 4q + \rho$, $0 \leqq \rho < q$. *If* $k = 3$, $e_4 \geqq q$ *or* $e_4 = q - 1$ *according as* $\rho > 0$ *or* $\rho = 0$. *If* $k = 4$ *or* 5, $e_4 \geqq q$ *when* $2\rho > q$, *but when* $2\rho \leqq q$, $E_4(A) = q - 1$ *at least for* $A \leqq 200t + 185q$.

Theorems 35-38 may be combined into

THEOREM 39. *Let* $k \geqq 2$, $m = t + q$, $1 < q < t$. *Then* $e_4 = q - 1$ *if and only if* $k = 2$, $t \leqq 3q$; $k = 3$, $2t \leqq 7q$ *or* $t = 4q$; $k = 4$ *or* 5, $2t \leqq 9q$; $k \geqq 6$, $t \leqq 4q$, *but with no conclusion when* $4q < t < 5q$. *In all further cases (including* $t \geqq 5q$ *if* $k \geqq 6$), $4 + e_4 \geqq L$ *of Theorem* 33.

41. THEOREM 40. *If* $k = 2$, 3 *or* 4, $m = q + t$, $0 < q < t$, *then* $2 + e_2$ *exceeds* L *of Theorem* 33.

If $k = 2$, there is a gap $\tau = 2t + 3q$ from $13t + 7q$ to $15t + 10q$. Then $2 + E_2 = \tau + 1$. If $q = 1$, $\tau + 1 > L$. Let $q > 1$. Since

$$\tau + 1 - (p + q - 1) = (p + 2)(q - 1) + 4 + 2\rho > 0,$$

the theorem follows if $p \geqq 6$. Also $\tau + 1 > q + 4$.

If $k = 3$, there is a gap $3t + 3q > \tau$ from $73t + 57q$ to $76t + 60q$. If $k = 4$, there is a gap τ from $133t + 112q$ to $135t + 115q$.

THEOREM 41. *If* $k = 2$ *or* 3, $m = q + t$, *then* $3 + e_3$ *is equal to or greater than the least* $s + e_s$ *for* $s \geqq 4$.

First, let $k = 2$. There is a gap from $20t + 13q$ to $21t + 12q$ and $\sigma = 3 + E_3 = t - q + 2$. If $q = 1$, $t \geqq 4$, $\sigma \geqq L$. If $q = 1$, $t = 3$, then $\sigma = 4 = 4 + e_4$ by Theorem 34. If $q = 1$, $t = 2$, there is a gap from $17t + 16q = 50$ to $19t + 14q = 52$ and $3 + E_3(51) = 4 = 4 + e_4$. Henceforth, let $q > 1$. Since

$$\sigma - (p + q - 1) = (p - 2)(q - 1) + \rho + 1,$$
$$\sigma - (q + 4) = q(p - 2) + \rho - 2,$$

* If $3\rho > 2q$, the earliest gap $> q$ is somewhere between $40t + 43q$ and $42t + 35q$. This holds also if $k = 5$.

$\sigma \geqq L$ unless $p = 1$ or 2. Let $t = q + \rho$, $0 < \rho < q$. Then $C = 28t + 21q$ is the only possible entry between $A = 28t + 18q$ and $B = 30t + 18q$. If $q \geqq 2\rho$, then $C \geqq B$ and there is a gap $2t$ from A to B, while $3 + E_3 = 2t + 2 \geqq q + 4$. But if $q < 2\rho$, the gap is from A to C, and $3 + E_3 = 3q + 2 \geqq q + 4$. Finally, let $t = 2q + \rho$. Then $\gamma = 19t + 11q$ is the only possible entry between $\alpha = 17t + 16q$ and $\beta = 19t + 14q$. According as $q \geqq 2\rho$ or $q < 2\rho$, there is a gap $2q + 2\rho$ from α to β or $3q$ from γ to β, and these gaps are the same as before.

Second, let $k = 3$. For every k there is a gap from $3q$ to t if $t > 3q$. Write $\epsilon = 3 + E_3 = t - 3q + 2$. If $q = 1$, $t \geqq 6$, then $\epsilon = L$. If $q > 1$, $p \geqq 5$, then $\epsilon \geqq L$ since $\epsilon - (p + q - 1) = (p - 4)(q - 1) + \rho - 1 \geqq 1$ for $p \geqq 6$.

When $3q \leqq t \leqq 5q$, there is a gap $6q - t$ from $52t + 42q$ to $51t + 48q$. Write ϕ for $3 + E_3 = 6q - t + 2$. If $q = 1$, $t = 3$ or 4, $e_4 = 0$ by Theorem 34 and $\phi = 8 - t \geqq 4 + e_4$, while if $t = 5$ there is a gap 3 from $3t + 3q = 18$ to $3t + 6q = 21$ and $3 + 2 = L$. Next, let $q > 1$. Since $p < 5$, $t < 5q$. Hence $\phi \geqq q + 4$ unless $5q = t + 1$, and then there is a gap $2q$ from $2t + q$ to $2t + 3q$, while $2 + 2q \geqq q + 4$.

Let $t = 2q + \rho$, $0 \leqq \rho < q$. Then $\gamma = 19t + 11q$ is the only possible entry between $\alpha = 17t + 16q$ and $\delta = 18t + 16q$. If $q \geqq 2\rho$, then $\gamma \leqq \alpha$ and there is a gap from α to δ. Then $3 + E_3 = t + 2 \geqq q + 4$ unless $\rho = 0$. But for $t = 2q$, $e_4 = q - 1$ and $t + 2 \geqq 4 + e_4$. But if $q < 2\rho$, there is a gap $5q - t$ from γ to δ and $3 + E_3 = 3q - \rho + 2 \geqq q + 4$.

There remains only the case $t = q + \rho$, $0 < \rho < q$. Then $e_4 = q - 1$ and $K = 51t + 48q$ is the only possible entry between $G = 55t + 39q$ and $H = 56t + 40q$. If $3q \geqq 5\rho$, then $K \geqq H$ and the gap is from G to H, while $3 + E_3 = 2 + 2q + \rho \geqq 4 + e_4$. But if $3q < 5\rho$, there is a gap from G to K and $3 + E_3 = 2 + 5q - 4\rho \geqq 4 + e_4$.

We do not investigate the case $k \geqq 4$, not having a table of sums by three expressed in terms of q and ρ.

Démonstration de Quelques Propriétés des Ensembles Abstraits.

PAR MAURICE FRÉCHET.

Introduction. Dans mon livre sous presse: *Les Espaces Abstraits et leur Théorie considérée comme Introduction à l'Analyse Générale* * je n'ai donné qu'un petit nombre des démonstrations, me contentant souvent de renvoyer aux différents mémoires où les démonstrations utilisées avaient été publiées. Cependant, parmi les propriétés citées dans mon livre, quelques unes étaient inédites, d'autres avaient été énoncées sans démonstration par certains auteurs, enfin plusieurs de ces propriétés n'avaient été démontrées que dans des cas particuliers. On trouvera ici les démonstrations de plusieurs de ces propriétés. Pour chacune d'elles, j'indiquerai entre parenthèses la page correspondante de mon livre (approximativement d'après la pagination du manuscrit).

La terminologie sera celle qui a été adoptée dans mon livre.† Les références bibliographiques seront indiquées par les chiffres en caractères gras, qui correspondent aux numéros figurant dans la liste bibliographique placée à la fin de mon livre.

Le mémoire traite des propriétés des ensembles appartenant:

1. à un espace (V), page 49; 2. à un espace accessible (H), page 59;
3. à un espace régulier, page 64; 4. à un espace (L), page 64;
5. à un espace (S), page 65; 6. à un espace (D), page 66.

1. *Propriétés des ensembles appartenant à un espace* (V).

I. *Tout ensemble E compact en soi au sens très strict possède la propriété de Borel* (page **270**). Il s'agit de montrer que si une famille dénombrable F d'ensembles $I_1, I_2, \cdots, I_n, \cdots$ couvre E, il en est de même d'une famille finie f convenablement extraite de F.

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* In 8° (25 x 16) x + 300 pages environ, chez Gauthier-Villars, Paris, 1928. Ouvrage faisant partie de la "Collection de monographies sur la Théorie des fonctions" publiée sous la direction d'Emile Borel.

† Ou dans mes derniers mémoires (sensiblement différente par conséquent de celle que j'avais d'abord employée dans ma Thèse).

Nous pourrons employer le mode de raisonnement utilisé par M. Borel pour le cas linéaire.

Ou bien l'ensemble I_1 couvre à lui seul E, et on peut constituer la famille f au moyen de I_1 seul; ou bien il existe au moins un point a_1 de E non intérieur à I_1. Le point a_1 est intérieur à l'un au moins des ensembles de F; soit $J_2 = I_{n_2}$, le premier d'entre eux, (J_2 est nécessairement distinct de I_1). Ou bien, on peut constituer f de $I_1, I_2, \cdots, I_{n_2}$, ou bien il existe au moins un point a_2 de E qui n'est intérieur à aucun des ensembles $I_1, \cdots, I_{n_2}$. Et ainsi de suite; supposons extraite de F une suite d'ensembles $I_{n_2}, \cdots, I_{n_p}$ et une suite de points de E distincts $a_1, \cdots, a_p$, telles que pour $k \leq p$, I_n soit le premier des ensembles de la famille F auquel a_{k-1} est intérieur et que a_k ne soit intérieur à aucun des ensembles $I_1, I_2, \cdots, I_{n_k}$. Ceci a lieu au moins pour $p = 2$. Alors a_p est intérieur à l'un au moins des ensembles de F; soit $I_{n_{p+1}}$ le premier; on a nécessairement $n_1 < n_2 < \cdots < n_p < n_{p+1}$. Ou bien on peut constituer f de $I_1, I_2, \cdots, I_{n_{p+1}}$. Ou bien, il existe au moins un point de E, a_{p+1}, qui n'est intérieur à aucun de ces derniers ensembles et qui est par suite distinct de $a_1, \cdots, a_p$.

Finalement, ou bien on peut constituer la famille finie f comme il était annoncé, ou bien la formation récurrente de la suite des points distincts $a_1, a_2, \cdots$ et des ensembles distincts $I_1, \cdots, I_{n_k}$ pourrait se poursuivre indéfiniment. Mais alors, le sous-ensemble $a_1 + a_2 + \cdots$ de l'ensemble E étant infini, il existerait un point x de E dont tout voisinage contient à son intérieur une infinité de points a_n distincts. Comme x est intérieur à l'un au moins des ensembles de F, soit I_r, il existerait un voisinage de x compris dans I_r et, par suite, I_r comprendrait à son intérieur une infinité de points a_s distincts. Or a_s n'est intérieur à aucun ensemble I_r tel que $r \leq n_s$; I_r ne peut comprendre à son intérieur que les points a_s tels que $n_s < r$, c'est-à-dire seulement un nombre fini d'entre eux.

II. *Tout ensemble E possédant la propriété de Borel est compact en soi* (page **270**). Autrement dit, pour tout sous ensemble infini e de E, il existe un point de E dont tout voisinage comprend une infinité de points de e.

Pour le démontrer, tirons de e une suite infinie, S, de points distincts $a_1, a_2, \cdots, a_n, \cdots$. S'il existait un point de E dont tout voisinage comprenne une infinité de points de S, les voisinages de ce point comprendraient aussi une infinité de points de e et la proposition serait établie.

Sinon, pour tout x de E, il existerait un voisinage de x ne comprenant qu'un nombre fini de points de S. Soit e_n l'ensemble des points x de E dont chacun desquels possède au moins un voisinage $V_x^{(n)}$ ne contenant aucun des points $a_n, a_{n+1}, \cdots$. Certains des e_n sont peut-être vides; mais tout x

appartient à l'un au moins des e_n. Pour tout e_n non vide, on peut former un ensemble I_n, somme des voisinages $V_x^{(n)}$ correspondant aux points x de e_n. L'ensemble E est couvert par la famille dénombrable des ensembles I_n. En vertu de l'hypothèse, il est donc aussi couvert par un nombre fini des I_n. Soit r le plus grand des indices de ceux-ci. Le point a_r appartenant à E appartiendrait à un ensemble I_n tel que $n \leq r$, il appartiendrait donc à un voisinage $V_x^{(n)}$ ne comprenant aucun des points $a_n, a_{n+1}, \cdots, a_r, \cdots$. D'où la contradiction annoncée.

Exemple de M. Sierpinski. Nous venons d'obtenir deux conditions, l'une suffisante, l'autre nécessaire pour qu'un ensemble ait la propriété de Borel. Aucune de ces deux conditions n'est à la fois nécessaire et suffisante. Pour montrer que la première condition n'est pas nécessaire, M. Sierpinski a bien voulu me communiquer l'exemple suivant. (Il sera curieux d'observer plus loin que cet exemple peut aussi servir à démontrer que la seconde condition n'est pas suffisante).

M. Sierpinski considère un espace (V) dont les éléments sont les nombres ordinaux de première et de seconde classe, et ne sont pourvus chacun que d'un seul voisinage. Le voisinage d'un entier a est constitué par a et par l'ensemble des nombres ordinaux de seconde classe. On prend comme voisinage d'un nombre α de seconde classe, l'ensemble des nombres ordinaux $\leq \alpha$. Il est alors clair que l'intérieur du voisinage V_a d'un entier a quelconque est réduit au point α, alors que l'intérieur du voisinage V_a d'un nombre de la seconde classe consiste dans l'ensemble des nombres de seconde classe appartenant à V_a.

L'espace considéré jouit de la propriété de Borel. Soit F une famille dénombrable d'ensembles $U_1, \cdots, U_n, \cdots$ qui couvre l'espace. L'un au moins des U, soit U_m, contient à son intérieur une infinité non dénombrable de points de cet espace puisque cet espace n'est pas dénombrable. Quelque soit le nombre ordinal α, il existe donc un nombre ordinal $\beta > \alpha$ et intérieur à U_m. Par suite le voisinage de β appartient à U_m. Donc U_m comprend tous les nombres $\leq \beta$ et en particulier les nombres $\leq \alpha$. Comme α est un point arbitraire de l'espace, on voit que tout l'espace appartient à U_m. Par suite tout l'espace est bien aussi intérieur à U_m, l'espace est couvert par le seul ensemble U_m extrait de F.

Puisque cet espace possède la propriété de Borel, il est compact en soi. Mais il ne l'est pas au sens très strict. En effet, extrayons de l'espace l'ensemble infini, e, constitué de tous les entiers. Il ne peut exister aucun point x de l'espace dont le voisinage contienne à son intérieur une infinité d'entiers. Car si x est un entier, l'intérieur de son voisinage se réduit au seul point x. Et

si x est un nombre de seconde classe, aucun entier b n'est intérieur au voisinage de x.

On peut encore utiliser ce même espace pour donner un exemple d'un ensemble E compact en soi sans posséder la propriété de Borel. Il suffit de prendre pour E l'ensemble constitué par tous les entiers et par ω, le premier nombre ordinal de seconde classe. Cet ensemble est couvert par la famille dénombrable constitué par les voisinages de ses points. Comme aucun de ces voisinages ne contient à son intérieur plus d'un point, il est impossible d'en extraire une famille finie couvrant E. E n'a pas la propriété de Borel. Par contre, si l'on considère un sous-ensemble infini, e, de E, le voisinage du nombre ω de E contient certainement une infinité de points de e. Donc E est compact en soi.

III. *La condition nécessaire et suffisante pour qu'un ensemble E soit compact (en soi) est que, quelle que soit la suite monotone de sous-ensembles (non vides) de E, à rangs entiers*

$$E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots$$

il existe au moins un point (de E) appartenant aux E_n ou appartenant à leurs dérivés (page **271**).

1°. Supposons E compact. On ne considère que des sous-ensembles E_n non vides: soit a_n un point de E_n.

Si l'un au moins des a_n est répété une infinité de fois c'est un point commun aux E_n.

Sinon la suite des a_n contient une suite S de points distincts. Alors il existe un point a (de E) dont tout voisinage contient une infinité de points distincts et appartenant à S. Par suite, aussi grand que soit n, tout voisinage de a contiendra au moins un point distinct de a et appartenant à E_n: le point a est commun aux dérivés E_n' des E_n.

2°. Montrons que la condition est suffisante. Soit e un sous-ensemble infini (s'il en existe) de E. Extrayons de e une suite infinie S de points distincts $a_1, a_2, \cdots$, et soit E_n l'ensemble des points $a_n, a_{n+1}, \cdots$. Aucun des E_n n'est vide et chacun contient le suivant. Il y a donc un point a (de E) commun aux E_n ou à leurs dérivés. Comme les E_n n'ont aucun point commun, on voit que dans tout voisinage de a, il y aura au moins un point, distinct de a, appartenant à E_n, quel que soit n. Il y aura donc dans tout voisinage de a, une infinité de points distincts appartenant à e.

IV. *Soit, dans un espace* (V), *une famille* F *d'ensembles fermés dont l'un au moins,* E, *est parfaitement compact. La condition nécessaire et suffisante pour qu'il existe au moins un point commun à tous les ensembles de* F *est que tout nombre fini d'ensembles de* F *possèdent un point au moins en commun* (page **271**).

La condition est évidemment nécessaire. Si elle est remplie, nous allons prouver qu'il est impossible que chaque point a de E soit disjoint de l'un au moins E_a des ensembles de F. En effet, dans ce cas, puisque E_a est fermé, il existerait au moins un voisinage V_a de a qui serait disjoint de E_a. La famille des ensembles V_a couvrant E, qui est parfaitement compact en soi, il existerait d'après la généralisation du théorème de Borel-Lebesgue par M. Chittenden, un nombre fini des V_a : $V_{a_1}, \cdots, V_{a_n}$, qui suffisent pour couvrir E. Les ensembles de F, $E_{a_1}, \cdots, E_{a_n}$, et aussi E, étant en nombre fini, auraient, par hypothèse, au moins un point commun α. Ce point α appartenant à E appartient à l'un des ensembles V_{a_i}, par exemple à V_{a_1}. Or les ensembles E_{a_1} et V_{a_1} sont disjoints, ils ne sauraient avoir le point α en commun.*

V. *Soit dans un espace* (V) *une famille* F *d'ensembles, dont l'un au moins est parfaitement compact et fermé. Si pour toute famille finie extraite de* F, *il y a au moins un point commun aux ensembles de cette famille finie, alors il y a sûrement un point qui est commun à tous les ensembles de* F *ou qui est commun à tous leurs dérivés* (page **272**).

Cette proposition généralise en partie le théorème III. Sa démonstration sera obtenue plus loin à propos d'un énoncé distinct de celui-ci (XXIV).

VI. *Si deux ensembles mutuellement connexes* E, F *sont chacun connexes, leur somme est connexe* (page **267**).

Soit $E + F = G + H$, il faut démontrer que si G, ni H ne sont vides, on a $GH' + G'H \neq 0$.

Or $E = E \cdot G + E \cdot H = G_1 + H_1$; $F = F \cdot G + F \cdot H = G_2 + H_2$.

Si G_1, ni H_1 ne sont vides alors G_1 et H_1 doivent être mutuellement connexes, (puisque E est connexe), donc G et H, qui les contiennent respectivement, sont aussi mutuellement connexes. On peut refaire le même raisonnement quand G_2, ni H_2 ne sont vides.

*** Cette proposition a été énoncée pour la première fois dans le cas des espaces euclidiens par M. F. Riesz. Elle a été retrouvée et sa démonstration publiée pour la première fois par M. Sierpinski (CIII). On verra en comparant sa démonstration avec la nôtre, qu'en passant du cas euclidien au cas infiniment plus général des espaces (V), bien loin d'avoir eu à compliquer sa démonstration, nous avons pu la simplifier.**

Reste le cas où l'un des ensembles G_1, H_1 est vide et où l'un des ensembles G_2, H_2 est vide. Si par exemple $G_1 = 0$, $G_2 = G \neq 0$; on a donc le cas où $H_2 = 0$ et alors $H_1 = H \neq 0$. Dès lors $E = H_1 = H$ et $F = G_2 = G$, G et H sont encore mutuellement connexes, comme E et F.

VII. *Si des ensembles deux à deux mutuellement connexes sont chacun connexe, leur somme est connexe* (page **267**). En effet, dans ce cas, d'après le théorème précédent, deux points quelconques de la somme appartiennent à un sous-ensemble connexe de cette somme. Cela suffit (page **267**) pour prouver que cette somme est connexe.

VIII. *Si un ensemble F est fermé, chacun de ses composants est un continu ou est réduit à un seul point* (page **269**).

Tout composant f d'un ensemble F étant connexe ou réduit à un point, il suffit de prouver que si F est fermé, ses composants sont aussi fermés. Si f n'était pas fermé, il existerait un point a de f' n'appartenant pas à f. Mais a appartiendrait à F', donc à F qui est fermé. L'ensemble $f + a$ est connexe, d'après l'avant-dernière proposition. Ce serait donc un sous-ensemble de F qui serait connexe et contiendrait f, c'est-à-dire contiendrait le plus grand sous-ensemble de F qui est connexe et contient un point de f.

IX. Par définition de l'intérieur I et de l'extérieur E d'un ensemble G, aucun point de I ne peut appartenir au dérivé du complémentaire de G, donc ne peut appartenir au dérivé de E. De même, aucun point de E ne peut appartenir à I'. Donc I et E ne peuvent être mutuellement connexes.

On va en conclure qu'*un ensemble ouvert et connexe I est un composant de l'ensemble complémentaire H de la frontière F de l'ensemble donné I* (page **267**).

Le plus grand sous-ensemble J de H qui est connexe et contient un point de I contient I, puisque I est un sous-ensemble connexe de H. Ainsi, J contient I. Si I n'était pas un composant de H, J contiendrait I plus un sous-ensemble non vide e de l'extérieur E de I. Il faudrait donc que I et e et par suite I et E fussent mutuellement connexes contrairement à la remarque ci-dessus.

X. Un ensemble G est connexe en un de ses points a, si dans tout voisinage V_a de a, il existe un voisinage W_a de a tel que deux points quelconques b, c de $G \cdot W_a$ appartiennent à un sous-ensemble connexe C de $G \cdot V_a$. On peut sans diminuer la généralité prendre $c = a$. Alors C est un sous-ensemble, connexe et comprenant a, de G. Donc C est un sous ensemble du composant K de G comprenant a. Un point b quelconques de $G \cdot W_a$ appartient ainsi,

avec a, à un sous-ensemble connexe C de $K \cdot V_a$: K est connexe en a. La somme S des ensembles connexes C qui ont tous en commun a, est un ensemble connexe appartenant à G. De plus S contient a, est contenu dans V_a et contient $G \cdot W_a$. En particulier, si a est un point intérieur à G, on peut choisir pour V_a un sous-ensemble convenable de G. Et alors, S contenant tout W_a, on voit que a est intérieur à S, donc intérieur au composant de G qui contient a.

Ainsi lorsqu'un ensemble G est connexe en un de ses points intérieurs, soit a, a est aussi intérieur au composant de G contenant a.

Si maintenant G est localement connexe, c'est-à-dire connexe en chacun de ses points, chaque composant de G est aussi localement connexe; et si G est ouvert, tout point a de G est intérieur au composant de G contenant a. Par suite tout composant de G est ouvert. Il est d'autre part aussi localement connexe comme G. Ainsi, *tout composant d'un ensemble ouvert et localement connexe est lui-même ouvert et localement connexe* (page **269**).

XI. Supposons l'ensemble G précédent séparable. Il existe un suite dénombrable N de points $a_1, a_2, \cdots$ de G telle que G appartienne à $N + N'$. Si a est un point de G appartenant à N', comme a est intérieur à G, a est intérieur au composant, γ, de G contenant a, donc il y a au moins un point a_n de N appartenant à γ. Par suite γ est identique au composant γ_n de G contenant a_n. Ainsi, *les composants d'un ensemble séparable, ouvert et localement connexe forment une famille dénombrable* (page **272**[bis]).

XII. Soit une famille F d'ensembles, dont les intérieurs existent et appartiennent à un même ensemble parfaitement séparable S. Lorsque ces intérieurs sont disjoints deux à deux, la famille F est dénombrable (page **275**).

En effet; par hypothèse, on peut prendre les voisinages des points de S parmi les ensembles d'une certaine suite dénombrable d'ensembles V_1, $V_2, \cdots, V_n, \cdots$.

Tout ensemble E de F a au moins un point intérieur a. Donc l'un des voisinages de a est un ensemble V_n appartenant à E. Soit $r(E)$ le rang du premier des ensembles $V_1, V_2, \cdots$ qui appartient à E. Pour deux ensembles distincts E, G appartenant à F, $r(E) \neq r(G)$, autrement les intérieurs de E, G ne seraient pas disjoints. Les ensemble E de F correspondent donc respectivement à des entiers $r(E)$ distincts. Donc F est dénombrable.

En particulier, *une famille de sous-ensembles ouverts et disjoints deux à deux, d'un ensemble parfaitement séparable est nécessairement dénombrable* (page **275**).

XIII. Les sous-ensembles ouverts d'un ensemble parfaitement séparable S sont chacun des sommes d'une famille d'ensembles $V_{n_1}, V_{n_2}, \cdots$ choisis dans la famille dénombrable $V_1, V_2, \cdots$. Chacun de ces sous-ensembles étant ainsi déterminé par une suite d'entiers, *la famille des sous-ensembles ouverts d'un ensemble parfaitement séparable S a au plus la puissance du continu* (page **275**).

XIV. Si F est un sous-ensemble fermé de S, $S—F$ est la partie commune à S à une des sommes $V_{r_1} + V_{r_2} + \cdots$. Donc: *la famille des sous-ensembles fermés d'un ensemble parfaitement séparable a au plus la puissance du continu* (page **275**).

XV. *Tout ensemble G parfaitement séparable est séparable* (page **279**[bis]). Appelons en effet N l'ensemble dénombrable des points a_n obtenus en choisissant arbitrairement un point a_n dans chacun des voisinages V_n des points de G. On voit que G appartient à $N + N'$

XVI. *Fonctionnelles et transformations ponctuelles.* Nous avons défini (page **202**) la continuité des transformations ponctuelles de deux façons; nous allons démontrer qu'elles sont équivalentes.

Disons qu'une transformation $T(E) = F$ est continue au point a de E, si pour tout voisinage V_b du transformé b de a, il existe un voisinage V_a de a tel que $T(E \cdot V_a)$ appartienne à $F \cdot V_b$. Supposons maintenant qu'il existe un sous-ensemble, e, de E ayant a pour point d'accumulation. V_a contient au moins un point a' de e distinct de a. Par suite V_b contiendra le point b' transformé de a'. Ce point b appartient au transformé f de e. Donc: ou bien b' est confondu avec b ce qui nécessite que b appartienne à f, ou bien V_b qui est un voisinage arbitraire de b, contient un point de f distinct de b. Ainsi: *si $T(E) = F$ est une transformation continue au point a, quelque soit le sous-ensemble e de E ayant a pour point d'accumulation, le point b, transformé de a, appartient nécessairement, soit au transformé f de e, soit à son dérivé f'. La réciproque est vraie* (page **202**). Autrement, il existerait un voisinage V_b de b tel que tout voisinage V_a de a contienne un point a' de E dont le transformé b' n'appartient pas à V_b. Alors b' est distinct de b et par suite a' de a. L'ensemble e des points a' a a pour point d'accumulation. Et pourtant, le point b n'appartient, ni au transformé f de e, ni à son dérivé f', puisque f est disjoint de V_b.

XVII. Considérons une fonctionnelle $U(x)$ uniforme sur un ensemble E compact (en soi). Quel que soit n, il existe un point a_m de E tel que $U(a_n) > M—1/n$, si $U(x)$ a sur E une borne supérieure finie M ou tel que $U(a_n) > n$

dans le cas contraire. Alors il existe un point a (de E) dans tout voisinage V_a duquel se trouvent une infinité des points a_n. Finalement, *il existe au moins un point a (de E) tel que la borne supérieure (finie ou non) de $U(x)$ sur l'ensemble E compact (en soi), soit égale à la borne supérieure de $U(x)$ sur la partie de E appartenant à un quelconque des voisinages de a* (page **277**).

XVIII. *Une fonctionnelle $U(x)$ continue sur un ensemble connexe C ne peut passer d'une valeur à une autre sans passer par toutes les valeurs intermédiaires* (page **277**). Soient a, b, deux points de C et α un nombre compris entre $U(a)$ et $U(b)$. Supposons, par exemple, $U(a) < \alpha < U(b)$. Si en aucun point x de C on n'avait $U(x) = \alpha$, on pourrait décomposer l'ensemble C en deux sous-ensembles disjoints et non vides

$$A \quad \text{où} \quad U(x) < \alpha,$$
$$B \quad \text{où} \quad U(x) > \alpha.$$

Donc A et B seraient mutuellement connexes, c'est-à-dire qu'il y aurait, par exemple, un point c de A qui appartient au dérivé B' de B. Alors, dans tout voisinage V_c de c, il existerait au moins un point x de B. On aurait $U(x) > \alpha$. Or on peut prendre le voisinage tel que $| U(x) - U(c) | < \epsilon$, ϵ etant un nombre positif arbitraire et on pourrait prendre $\epsilon < \alpha - U(c)$. On aurait donc $U(x) < \alpha$ sur certains points de B, contrairement à sa définition.

XIX. *La limite $U(x)$ d'une suite de fonctionnelles $U_n(x)$ continues en a relativement à E, qui converge uniformément sur E, est continue en a relativement à E* (page **280**).

Soit ϵ un nombre positif arbitraire. Par hypothèse, il existe un nombre p tel que pour $n > p$ $| U(x) - U_n(x) | < \epsilon/3$ en tout point x de E et il existe un voisinage de a sur lequel l'oscillation de $U_n(x)$ est inférieure à $\epsilon/3$. Alors pour deux points y, z de ce voisinage et de E, on a

$$| U(y) - U(z) | \leqq | U(y) - U_n(y) | + | U_n(y) - U_n(z) | + | U_n(z) - U(z) | < \epsilon.$$

XX. *La limite $U(x)$ d'une suite de fonctionnelles $U_n(x)$ également continues en a relativement à E et convergeant partout sur E est continue en a relativement à E* (page **281**).

Pour ϵ positif arbitraire, il existe par hypothèse un voisinage V_a de a sur lequel l'oscillation de chaque U_n est inférieure à $\epsilon/3$. D'autre part, pour n

assez grand ($n > n_x$), on a $| U(x) - U_n(x) | < \epsilon/3$. Pour deux points quelconques y, z de V_a, prenons $n > n_y$ et $n > n_z$; on aura

$$| U(y) - U(z) | \leqslant | U(y) - U_n(y) | + | U_n(y) - U_n(z) | + | U_n(z) - U(z) | < \epsilon.$$

XXI. *Si, de plus, E est compact en soi et si la suite est également continue en chaque point de E, la convergence est nécessairement uniforme sur E* (page **281**).

En effet, dans le cas contraire, il y aurait un nombre $\epsilon > 0$ et quel que soit p, un entier $n_p > p$ et un point x_p de E tels que

$$| U(x_p) - U_{n_p}(x_p) | > \epsilon.$$

Si parmi les x_p, il n'y avait qu'un nombre fini de points distincts, une infinité des x_p seraient identiques à un même point a. On aurait donc $| U(a) - U_{n_p}(a) | > \epsilon$ avec $n_p > p$, ce qui est impossible puisque la suite converge en chaque point a de E. Dans le cas contraire, on peut ne retenir que les x_p distincts et alors, comme E est compact en soi, il existe un point a de E dans tout voisinage duquel se trouve une infinité des x_p. D'autre part, il existe un voisinage V_a de a où l'oscillation de tous les U_n est inférieure a $\epsilon/3$ et où, par suite, celle de U est $\leq \epsilon/3$. Pour une infinité de valeurs de p, on aura donc

$$| U(a) - U(x_p) | \leq \epsilon/3, \quad | U_{n_p}(a) - U_{n_p}(x_p) | < \epsilon/3;$$

et pour p assez grand

$$| U(a) - U_{n_p}(a) | < \epsilon/3$$

On aurait donc $| U(x_p) - U_{n_p}(x_p) | < \epsilon$ pour une infinité de valeurs de p *contrairement à l'hypothèse.*

XXII. *La composition de deux transformations continues fournit une transformation continue* (page **283**). Soient deux transformations $y = A(x)$ et $z = B(y)$. Supposons $A(x)$ continue au point a relativement à un ensemble E; supposons que $B(y)$ soit continue au point $b = A(a)$ relativement à l'ensemble F transformé de E par A. Il s'agit de démontrer que la transformation $z = B[A(x)]$ est continue au point a relativement à E. Soit en effet $c = B(b)$. Les ensembles E, F et l'ensemble G, transformé de F par $B(x)$ appartiennent respectivement par hypothèse à trois espaces (V), distincts ou non. Soit V_c un voisinage arbitraire de c; il existe, par hypothèse, un voisinage V_b de b tel que le transformé par $B(x)$ de $F \cdot V_b$ appartienne à V_c. Il existe de même un voisinage V_a de a tel que le transformé par $A(x)$ de $E \cdot V_a$ appar-

tienne à V_b. Donc il existe un voisinage V_a de a tel que le transformé par $B[A(x)]$ de $E \cdot V_a$ appartienne à V_c.

XXIII. *Toute transformation continue A transforme un ensemble connexe E en un ensemble F connexe* (page **283**).

Soit $F = H_1 + H_2$. Tout point de E est transformé, soit en un point de H_1, soit en un point de H_2; appelons G_1 le sous-ensemble des point de E transformés chacun en un point de H_1, $G_2 = E - G_1$. Si nous supposons H_1 et H_2 disjoints et non vides, G_1 et G_2 seront disjoints et non vides. Par suite $G_1 \cdot G_2' + G_2 \cdot G_1'$ n'est pas vide. Or le transformé de cet ensemble appartient à $H_1 \cdot (H_2 + H_2') + H_2 \cdot (H_1 + H_1') = H_1 \cdot H_2' + H_2 \cdot H_1'$, puisque H_1 et H_2 sont disjoints. Donc $H_1 \cdot H_2' + H_2 \cdot H_1'$ n'est pas vide.

Il faut observer que le raisonnement ne s'étend pas au cas où E serait seulement supposé bien enchaîné. Car un point a appartenant à $G_1 \cdot G_2' + G_2 \cdot G_1' + G_1' \cdot G_2'$ pourrait ne pas appartenir à E et alors on ne peut plus parler du transformé de a.

Non seulement la démonstration, mais l'énoncée ne s'étend pas. Par exemple, considérons la figure inverse D d'une circonférence S obtenue en prenant le pôle d'inversion p sur S. Prenons pour E un arc de S s'étendant de chaque coté de p mais privé du point p. Alors E est un ensemble non connexe mais bien enchaîné que l'inversion transforme dans un ensemble F composé de deux demi-droites situées sur D et séparées par un segment de D. L'ensemble F n'est ni connexe, ni bien enchaîné.

2. *Propriétés des espaces accessibles ou espaces* (H).

XXIII. *Pour un espace* (V) *satisfaisant aux condition* 1°, 2°, 3° *de F. Riesz* (page **206**), *il est équivalent de supposer que tout ensemble dérivé est fermé ou de supposer que l'intérieur de chaque ensemble est un ensemble ouvert* (page **247**).

[Dans l'un et l'autre cas, l'espace considéré sera donc un espace accessible ou espace (H)]. En effet, admettons la première condition et soit I l'intérieur d'un ensemble E. Si I n'était pas ouvert, il y aurait un point x intérieur à E sans être intérieur à I. C'est-à-dire qu'au moins un voisinage V_x de x appartiendrait entièrement à E et que tout voisinage de x contiendrait au moins un point y distinct de x et n'appartenant pas à I. L'ensemble des points y appartiendrait au complémentaire de I, donc au complementaire C de E ou à son dérivé C'. Et il aurait x pour point d'accumulation. Donc x appartiendrait à C' ou à C''. Or x appartenant à I ne peut appartenir à C'. Il y aurait donc un point x de C'' n'appartenant pas à C' contrairement à l'hypothèse.

Inversement, admettons la seconde condition. Si un ensemble dérivé F' n'était pas fermé il y aurait un point z de F'' n'appartenant pas à F'. Soit $f = F - z$; en vertu des trois conditions de F. Riesz, z sera aussi un point de f'' n'appartenant, ni à f', ni à f. Soit c l'ensemble complémentaire de f; z est intérieur à c. En vertu de la seconde condition, z est même intérieur à l'intérieur i de c. Il y a donc un voisinage V de z appartenant à i. Puisque z appartient à f'', il y aurait au moins un point t de V, distinct de z et appartenant à f'. Ce point t étant intérieur à c ne peut pourtant pas être point d'accumulation du complémentaire f de c.

XXIV. *Soit dans un espace accessible, ou espace* (H), *une famille* F *d'ensembles dont l'un au moins,* E, *est parfaitement compact et fermé. La condition nécessaire et suffisante pour qu'il existe au moins un point commun à tous les ensembles de* F *ou au moins un point commun aux dérivés de ces ensembles est qu'il en soit de même pour toute famille finie extraite de* F (page **272**).

La condition est évidemment nécessaire. Nous allons montrer que si elle est remplie, il est impossible que, pour tout point a de E, il existe un ensemble e_a de F disjoint de a et un ensemble E_a de F (distinct ou non de e_a) dont a n'est pas point d'accumulation. En effet, dans ce cas, il existerait un voisinage V_a de a n'ayant avec E_a aucun point commun distinct de a. Alors les V_a couvriraient E et, puisque cet ensemble est compact et fermé, E pourrait être couvert par un nombre fini des V_a : $V_{a_1}, \cdots, V_{a_n}$. Alors, si on considère les ensembles de F: E, $e_{a_1}, \cdots, e_{a_n}$, $E_{a_1}, \cdots, E_{a_n}$, qui sont en nombre fini, il existerait un point α commun à ces ensembles ou commun à leurs dérivés. Comme E est fermé, α appartient dans les deux cas à E. Par suite, il est intérieur à un des V_{a_i}, par exemple à V_{a_1}.

Dans le premier cas, α appartenant à E_{a_1} et V_{a_1} qui n'ont aucun point commun distinct de a_1, α coinciderait avec a_1. Or cela est impossible puisque α appartient à e_{a_1} tandis que a_1 ne lui appartient pas.

Dans le second cas, α étant intérieur à V_{a_1}, un des voisinages W_a de α appartient à V_{a_1}, et comme α est point d'accumulation de E_{a_1}, il y aurait un point au moins β de E_{a_1}, distinct de α, en commun avec W_a, donc avec V_{a_1}. Comme E_{a_1} et V_{a_1} n'ont en commun aucun point distinct de a_1, β coincide avec a_1. Puisque $E_{a_1} = a_1 + (E_{a_1} - a_1)$ a α pour point d'accumulation et puisque l'espace envisagé est accessible, α serait point d'accumulation de $E_{a_1} - a_1$ ou de l'ensemble réduit à a_1. Or $E_{a_1} - a_1$ étant disjoint de V_{a_1}, donc de W_a, ne peut avoir α pour point d'accumulation. Il faudrait donc que le point α fut point d'accumulation d'un ensemble composé d'un seul point, ce qui est impossible dans un espace accessible.

Il y a lieu de remarquer que la partie de la démonstration relative au premier cas reste valide pour un espace (V). Nous obtenons donc ici la démonstration du théorème V.

XXV. *La frontière F d'un ensemble E est un ensemble fermé* (page **286**). En effet, si C est l'ensemble complémentaire de E, on a, par définition,

$$F = C \cdot E' + C' \cdot E.$$

Dans un espace accessible, la dérivation est distributive, donc:

$$F' = C' \cdot E'' + C''E' + C'E';$$

et puisque tout ensemble dérivé est fermé,

$$F' \subset C'E' = C' \cdot E' \cdot E + C' \cdot E' \cdot C \subset C'E + E'C = F.$$

XXVI. *Tout ensemble séparable E appartient à un ensemble fermé séparable, par exemple à la fermeture $E + E'$ de E* (page **286**).

Il existe, par hypothèse, un sous-ensemble dénombrable N de E tel que

$$N \subset E \subset N + N'.$$

D'où

$$N' \subset E' \subset N' + N'' = N'.$$

Donc $E' = N'$, d'où $E + E' = N + N'$. Alors

$$E \subset N + N' \subset E + E'.$$

Et l'ensemble $E + E'$ est fermé et puisque $N \subset E + E' = N + N'$, $E + E'$ est aussi séparable.

XXVII. La formule $E' = N'$ prouve aussi que *tout ensemble séparable dense en soi appartient au dérivé de l'un de ses sous-ensembles dénombrables et que tout ensemble séparable parfait est le dérivé de l'un de ses sous-ensembles dénombrables* (page **284**).

XXVIII. *Dans un espace accessible, tout ensemble parfaitement séparable et non clairsemé contient un ensemble dénombrable dense en soi* (page **274**).

L'ensemble E n'étant pas clairsemé contient un ensemble D dense en soi. Comme E est parfaitement séparable, D est aussi parfaitement séparable et par suite séparable (XV). Or tout ensemble séparable et dense en soi, D, appartient au dérivé d'un de ses sous-ensembles dénombrables (XXVII) soit N. Et comme $N \subset D \subset N'$, N est dense en soi.

XXIX. *Dans un espace accessible, tout ensemble parfaitement séparable E a au plus la puissance du continu* (page **285**). En effet, par hypothèse,

il existe une famille dénombrable d'ensembles $V_1, V_2, \cdots, V_n, \cdots$ telle que la famille des voisinages de tout point x de E soit équivalente à la famille constituée de ceux des ensembles V_n, (soient $V_{n_1}, V_{n_2}, \cdots$) auxquels x est intérieur. Il n'y a d'ailleurs, puisque l'espace est accessible, qu'un point x commun aux voisinages $V_{n_1}, V_{n_2}, \cdots$ de x.

Tout point x détermine une suite d'entiers $n_1, n_2, \cdots$ telle que x soit intérieur aux ensembles $V_{n_1}, V_{n_2}, \cdots$ et à aucun autre des V_n.

Réciproquement, une telle suite d'entiers, c'est-à-dire une suite telle qu'il existe au moins un point x intérieur à $V_{n_1}, V_{n_2}, \cdots$ et à aucun autre des V_n détermine un point x de E *et un seul.*

Les points de E correspondent donc de façon biunivoque à certaines suites d'entiers $n_1, n_2, \cdots$.

Donc E a au plus la puissance du continu.

XXX. Dire qu'un point a d'un espace accessible est *à caractère dénombrable,* c'est dire qu'on peut supposer dénombrable la famille des voisinages de a. Soient donc $W_1, W_2, \cdots, W_n, \cdots$ ceux-ci. Appelons V_2 le premier des W_n qui est commun à W_1 et W_2. En posant $V_1 = W_1$, appelons en général V_p le premier des W_n qui est commun à $V_1, \cdots, V_{p-1}$ et W_p. On voit qu'on aura

$$V_1 \supset V_2 \supset \cdots \supset V_p \supset \cdots .$$

Comme les W, ces ensembles V ne pourront avoir en commun que le point a. Leur famille est équivalent à celle des W. Car chaque V est un W et chaque W_p comprend un V_p. Ainsi, *si un point a d'un espace accessible est à caractère dénombrable, on peut supposer que la famille de ses voisinages est une suite dénombrable d'ensembles V_p n'ayant en commun que le point a et dont chacun contient le suivant* (page **298**).

C'est en particulier ce que a lieu si a est un point quelconque d'un ensemble parfaitement séparable.

(Dans le cas exceptionel où la suite des V_p serait finie, la famille des voisinages de a serait même équivalente à la famille constituée par un seul ensemble: le dernier des V_p).

XXXI. *Toute transformation continue transforme un ensemble* (*parfaitement*) *compact en soi en un ensemble* (*parfaitement*) *compact en soi* (page **288**).

En effet, soit A une transformation continue d'un ensemble E d'un espace accessible en un ensemble E_1 d'un espace accessible (distinct ou non du premier). Soit e_1 un sous-ensemble infini de E_1. Chaque point y de e_1 est le transformé d'au moins un point x de E. Ne prenons qu'un point x de E pour

chaque point y de e_1. Nous formerons ainsi un ensemble e de points distincts; e est un sous-ensemble infini de E et son transformé est e_1.

Si E est (parfaitement) compact en soi, il existera un point a de E dont tout voisinage comprend au moins un point de e distinct de a (comprend un ensemble de points de e ayant même puissance que e). Or si V_{a_1} est un voisinage arbitraire du transformé a_1 de a, il existe un voisinage V_a de a, tel que le transformé de $e \cdot V_a$ appartienne à V_{a_1}. Donc V_{a_1} comprend au moins un point de e_1 distinct de a_1 (comprend un ensemble de points de e_1 ayant une puissance au moins égale à celle de $e \cdot V_a$, c'est-à-dire à celle de e, c'est-à-dire à celle de e_1 et, comme la puissance de $e_1 \cdot V_{a_1}$ est au plus égale à celle de e_1, ces deux puissances sont égales).

XXXII. Démontrons une proposition qui constitue en quelque sorte une réciproque du théorème XXI.

Soit F une famille de fonctionnelles continues sur un ensemble E à caractère dénombrable (c'est-à-dire dont la famille des voisinages de chaque point de E est dénombrable). *Pour que de tout sous-ensemble infini de fonctionnelles de F, on puisse extraire une suite qui converge uniformément sur E, il faut qu'en chaque point de E, les fonctionnelles de F soient également bornées et également continues relativement à E* (page **288**).

1°. S'il existe au moins un point a de E où les fonctionnelles de F ne sont pas également bornées, c'est que, pour tout entier n, il existe une fonctionnelle $U_n(x)$ de F telle que $| U_n(a) | > n$. Cela est impossible car, par hypothèse, on pourrait extraire de la suite des $U_n(x)$ une suite qui converge uniformément sur E et en particulier qui converge en a. Cette démonstration est valable pour un ensemble E quelconque appartenant à un espace (V) quelconque. Il n'en est pas de même de la démonstration du second point.

2°. S'il existe un point a de E où les fonctionnelles de F ne sont pas également continues, il existe un nombre $\epsilon > 0$, tel que pour tout voisinage V de a, l'oscillation sur $E \cdot V$ de l'une au moins des fonctionnelles de F soit supérieure à ϵ. D'après ce qui précède, on peut supposer que les voisinages de a forment une suite dénombrable monotones d'ensembles $V_1, V_2, \cdots$ n'ayant en commun que le point a. Soit U_n une fonctionnelle de F dont l'oscillation sur V_n est supérieure à ϵ. On peut extraire par hypothèse de la suite des $U_n(x)$ une suite uniformément convergente sur E. Appelons $U(x)$ sa limite. Il y a un voisinage V_p de a sur lequel l'oscillation de $U(x)$ est $< \epsilon/6$. D'autre part, pour n assez grand, $| U_n(x) - U(x) | < \epsilon/6$ sur tout E. Par suite si x appartient à $E \cdot V_p$

$$| U_n(x) - U_n(a) | \leq | U_n(x) - U(x) | + | U(x) - U(a) | + | U(a) - U_n(a) | < \epsilon/2.$$

Donc l'oscillation de $U_n(x)$ sur V_p serait $< \epsilon$, pour p fixé, n assez grand. Si donc on prend n assez grand et en particulier $> p$, l'oscillation de $U_n(x)$ sur V_n serait $< \epsilon$, contrairement à l'hypothèse.

3. *Espace régulier.*

XXXIII. *Dans un espace régulier* (page **240**), *tout ensemble ouvert parfaitement séparable, E, est la somme d'une famille dénombrable d'ensembles fermés* (page **299**).

L'ensemble E étant parfaitement séparable, soit S la suite dénombrable des voisinages $V_1, V_2, \cdots, V_n, \cdots$ des points de E. Pour chaque point a, l'un au moins des voisinages V_n auxquels a est intérieur appartient a E. Ainsi on peut couvrir E au moyen d'une famille f d'ensembles V_n appartenant chacun à E. Donc E est la somme des ensembles de cette famille. D'autre part, dire que l'espace est régulier, c'est dire qu'on peut supposer fermés les ensembles de S. La famille f est une famille dénombrable d'ensembles fermés.

4. *Propriétes des espaces* (L).

XXXIV. Nous avons appelé dans notre Thèse ensemble compact (en soi) de points d'un espace (L), un ensemble E dont tout sous-ensemble infini e donne lieu à au moins un point d'accumulation a (appartenant à E). Il y aura donc une suite de points de e: $a_1, a_2, \cdots, a_n, \cdots$ distincts et convergeant vers a. Comme a est aussi point limite de la suite $a_n, a_{n+1}, \cdots$, il y aura dans tout voisinage de a une infinité de points de e distincts. De sorte que E est aussi compact (en soi) au sens adopté dans notre livre pour les ensembles de points d'un espace (V).

Dans un espace (V) quelconque, un ensemble (parfaitement) compact et fermé est (parfaitement) compact en soi. Dans un espace (L), *la réciproque est vraie* (page **301**). En effet, dans un tel espace tout point d'accumulation a d'un ensemble E est point limite d'une suite convergente S extraite de E et formée d'éléments distincts. Si E est compact en soi (et, à fortiori, si E est parfaitement compact en soi), il existe un point de E qui est point d'accumulation de S. Comme a est le seul point d'accumulation de S, on voit que a appartient à E.

XXXV. *La condition nécessaire et suffisante pour qu'un ensemble E soit compact est, que pour toute infinie monotone de sous-ensembles fermés de E*

$$E_1 \supset E_2 \supset E_3 \supset \cdots \supset E_n \supset \cdots,$$

il existe au moins un point commun aux ensembles de cette suite (page **301**).

La condition est nécessaire, cela résulte du théorème III (ou **22**, p. 7). La condition est suffisante; en effet si E n'était pas compact, il existerait un sous-ensemble infini, e, de E, n'ayant aucun point d'accumulation. Soit, S, une suite infinie de points de e, distincts

$$a_1, a_2, \cdots, a_n, \cdots$$

et E_n l'ensemble des points $a_n, a_{n+1}, \cdots$. Il n'existe aucun point d'accumulation de E_n. La suite des E_n serait donc une suite monotone de sous-ensembles fermés de E. Il y aurait donc un point commun aux E_n, ce qui est évidemment impossible.

XXXVI. *Toute transformation continue d'un ensemble compact et fermé E appartenant à un espace* (L) *en un ensemble F appartenant à un espace* (L)*—distinct ou non du premier—fournit un ensemble compact et fermé* (pages **44** et **303**).

En effet, soit f un sous-ensemble infini de F. Comme au paragraphe XXXVI, on peut le considérer comme transformé d'un sous-ensemble infini e de E choisi de sorte qu'un point de f corresponde à un seul point de e. Il y a un point a de E qui est point d'accumulation de e. Le transformé b de a est peut-être aussi transformé d'un point a_1 de e distinct de a. Mais a est point d'accumulation de $e - a - a_2$. Alors b, ne pouvant appartenir à f-b transformé de $e - a - a_1$, sera point d'accumulation de f-b, donc de f. Et b appartient à F.

Donc f a au moins un point d'accumulation appartenant à F. Ainsi F est compact en soi, c'est-à-dire, d'après XXXIV, compact et fermé.

5. *Propriété des espaces* (S).

XXXVII. M. Hausdorff a démontré (X, page **265**) que tout espace de Hausdorff à caractère dénombrable est un espace (S).

La réciproque est vraie (page **248**). Comme tout espace (S) est un espace accessible, il suffit de démontrer qu'un espace (S) à caractère dénombrable possède la porpriété D) de M. Hausdorff. Dire qu'il ne la possède pas, ce serait dire qu'il existe deux points a, b distincts ne possédant pas deux voisinages disjoints. Soient $V_a^1, V_a^2, \cdots$; $V_b^1, V_b^2, \cdots$ les voisinages respectifs de a et de b. Il y aurait un point a_n commun à V_a^n et V_b^n quel que soit n. Comme a_n ne peut-être identique à la fois à a et b, l'un au moins des deux points a et b, a par exemple, est distinct d'un infinité des a_n. D'ailleurs, on peut supposer (voir XXX) que la suite $V_a^1, V_a^2, \cdots$ est monotone, de même que la suite $V_b^1, V_b^2, \cdots$. Donc, a, par exemple, est point d'accu-

mulation de l'ensemble des a_n. C'est, par conséquent, la limite unique d'une certaine suite convergente de points distincts $a_{n_1}, a_{n_2}, \cdots$ extraite de la suite des a_n. Il ne peut donc y avoir dans cette suite qu'un nombre fini ou nul de points coincidant avec b. La suite formée par les autres points est formée de points distincts de a et de b, figurant dans une infinité des V_a^n et une infinité des V_b^n et par conséquent figurant dans tous les voisinages de a et tous les voisinages de b. C'est donc une suite convergeant vers a et qui aurait aussi b pour point d'accumulation.

6. *Propriétés des espaces* (D).

XXXVIII. *Tout ensemble complet E fait partie d'un ensemble fermé complet G extrait de ce même espace* (page **308**).

Nous appelons ensemble complet un ensemble E tel que sans altérer la convergence ou la non-convergence des suites de points de E, ni leurs limites, on puisse définir celles-ci par l'intermédiaire de distances vérifiant le critère de convergence de Cauchy. Autrement dit: une certaine distance $((x, y))$ étant définie pour tout couple de points x, y de l'espace (D) considéré, nous supposons qu'on peut en outre attacher à tout couple de points x, y de E une certaine distance (x, y) égale au non à $((x, y))$. Comme la distance $((x, y))$, cette distance (x, y) doit être ≥ 0, nulle seulement si x et y sont confondus, telle que

$$(x, y) \leq (x, z) + (y, z)$$

quels que soient x, y, z de E; telle que si y de E tend vers x de E, (x, y) tend vers zéro et réciproquement. Mais, en outre, elle doit être telle que, si, pour une suite $x_1, x_2, \cdots, x_n, \cdots$ de points de E, les distances (x_n, x_{n+p}) sont infiniement petites avec $1/n$, cette suite converge (sa limite appartenant ou non à E); et inversement.

Et nous nous proposons de montrer qu'on peut attacher à tout couple de points x, y de l'ensemble fermé $F = E + E'$, une distance $[x, y]$ égale à (x, y) quand x et y appartiennent à E et satisfaisant relativement à F aux mêmes conditions que nons venons de spécifier pour (x, y) relativement à E.

Si cela est possible, on aura

$$| [x, y] - (x_n, y_n) | \leq [x, x_n] + [y, y_n]$$

pour tout couple x, y de F et tout couple x_n, y_n de E. On pourra toujours prendre x_n et y_n de sorte qu'ils convergent respectivement vers x et y de F et, par suite, on aura:

$$(5) \qquad [x, y] = \lim_{x_n \to x, y_n \to y} (x_n, y_n)$$

Inversement, ne supposons plus le problème résolu et posons, par définition de $[x, y]$, la relation (5) pour tout couple de points x, y de F et toute suite de couples x_n, y_n de points de E convergeant respectivement vers x, y.

Pour que la définition soit légitime, il faut que le second membre de (5) ait sûrement une limite et que celle-ci ne dépende que de x et y. On a d'abord

$$| (x_n, y_n) - (x_{n+p}, y_{n+p}) | \leq (x_n, x_{n+p}) + (y_n, y_{n+p}).$$

D'après les hypothèses faites, le second membre est, indépendamment de p, infiniment petit avec $1/n$. La suite de nombres $(x_1, y_1), \cdots, (x_n, y_n), \cdots$ est donc bien convergente.

Sa limite ne dépend que de x et de y. Soit, en effet, x_n', y_n', un couple de points de E convergeant respectivement vers x, y. Alors le couple x, y sera aussi la limite de la suite de couples de E

$$x_1, y_1; x_1', y_1'; x_2, y_2; \cdots; x_n, y_n; x_n', y_n'; \cdots.$$

Donc la suite des distances

$$(x_1, y_1); (x_1', y_1'); (x_2, y_2); \cdots; (x_n, y_n); (x_n', y_n'); \cdots$$

d'après ce qui précède, est aussi convergente, ce qui prouve que

$$\lim_{x_n' \to x,\ y_n' \to y} (x_n', y_n') = \lim_{x_n \to x,\ y_n \to y} (x_n, y_n),$$

Ainsi $[x, y]$ est un nombre bien déterminé pour tout couple de points de F. De plus, si x, y appartiennent à E, on peut prendre $x_n \equiv x$, $y_n \equiv y$, ce qui montre que dans ce cas

$$[x, y] = (x, y).$$

Si maintenant x, y sont deux points quelconques de F, on a évidemment

$$[x, y] \geq 0, \qquad [x, x] = 0.$$

Inversement, si $[x, y] = 0$, on a $\lim (x_n, y_n) = 0$.

Or, par hypothèse (x_n, x_{n+p}) tend indépendamment de p vers zéro avec $1/n$. Donc la suite de points

$$x_1, y_1; x_2, y_2; \cdots; x_n, y_n; \cdots$$

vérifie aussi les conditions de Cauchy relativement à la distance $(\cdots)$: elle converge et alors $x \equiv y$.

On a pour trois points x_n, y_n, z_n, quelconques sur E

$$(x_n, y_n) \leq (x_n, z_n) + (z_n, y_n);$$

donc à la limite, si ces points tendent vers x, y, z de F,

$$[x, y] \leq [x, z] + [z, x].$$

Montrons aussi que si $y^{(p)}$ de F tend vers y de F, on a $\lim_{p\to\infty} [y^{(p)}, y] = 0$ et inversement.

Supposons d'abord que chaque $y^{(p)}$ appartienne à E. Alors le couple $y^{(p)}$, $y^{(p+n)}$ de E tend vers le couple $y^{(p)}$, y de F quand n croit indéfiniment. Donc

$$[y^{(p)}, y] = \lim_{n\to\infty} (y^{(p)}, y^{(p+n)}).$$

Puisque la suite des $y^{(p+n)}$ de E est convergente, les nombres $y^{(p)}$, $y^{(p+n)}$ sont infiniment petits avec $1/p$, indépendamment de n. Donc $[y^{(p)}, y]$ est infiniment petit avec $1/p$; on a bien dans ce cas

$$\lim_{p\to\infty} [y^{(p)}, y] = 0.$$

Passons au cas où $y^{(p)}$ appartient à F; il est limite d'une suite de points de E. On peut donc prendre, d'après ce qui précède, un point y_p de E tel que $[y_p, y^{(p)}] < 1/p$, et aussi $((y_p, y^{(p)})) < 1/p$. Comme $((y, y_p)) \leq ((y, y^{(p)})) + ((y^{(p)}, y_p))$, on voit que y_p de E tendra vers y de F. Donc

$$[y, y^{(p)}] \leqq [y, y_p] + [y_p, y^{(p)}] < [y, y_p] + 1/p.$$

Or

$$\lim_{p\to\infty} [y, y_p] = 0.$$

On a donc encore

$$\lim_{p\to\infty} [y, y^{(p)}] = 0.$$

Réciproquement, supposons que pour les points y, $y^{(p)}$ de F, on ait

$$\lim_{p\to\infty} [y, y^{(p)}] = 0.$$

Prenons des points z_p, $z^{(p)}$ de E; on peut, pour p fixe, les faire varier et les fair tendre vers y, $y^{(p)}$ respectivement. On peut donc, d'après ce qui précède, les prendre tels que $[z_p, y] < 1/p$; $[z^{(p)}, y^{(p)}] < 1/p$, et aussi

$$((z^{(p)}, y^{(p)})) < 1/p.$$

Or, on a

$$(z^{(p)}, z_p) = [z^{(p)}, z_p] \leq [z^{(p)}, y^{(p)}] + [y^{(p)}, y] + [y, z_p] < [y^{(p)}, y] + 2/p.$$

Donc

$$\lim_{p\to\infty} (z^{(p)}, z_p) = 0. \tag{6}$$

On a d'autre part,

$$(z^{(p)}, z^{(p+n)}) \leq (z^{(p)}, z_p) + (z_p, z_{p+n}) + (z_{p+n}, z^{p+n}).$$

En vertu de l'égalité (6) et de la convergence de la suite des z_p, on voit, d'après cette inégalité, que $(z^{(p)}, z^{(p+n)})$ est infiniment petit avec $1/p$ indépendamment de n. La suite des $z^{(p)}$ est donc aussi convergente. Soit z sa limite, qui est un point de F. On a

$$[z, y] \leq [z, z^{(p)}] + [z^{(p)}, z_p] + [z_p, y] < [z, z^{(p)}] + [z^{(p)}, z_p] + 1/p,$$

et les trois termes du troisième membre tendent vers zéro avec $1/p$: le deuxième terme en vertu de (6), le premier d'après la proposition directe démontrée à l'instant. Donc $[z, y] = 0$, et on a vu que ceci implique $z \equiv y$; par suite $z^{(p)}$ tend vers y.

Enfin, on a

$$((y, y^{(p)})) \leq ((y, z^{(p)})) + ((z^{(p)}, y^{(p)})) \leq ((y, z^{(p)})) + 1/p.$$

Donc, on a bien

$$y = \lim_{p\to\infty} y^{(p)}.$$

Il nous reste à démontrer que la distance $[x, y]$, qu'on vient de définir pour tout couple de points de F, se prête à la généralisation du critère de Cauchy. Si $x_1, x, \cdots$ est une suite convergente de points de F, sa limite x appartient à F; et on a

$$[x_p, x_{n+p}] \leq [x, x_p] + [x, x_{n+p}].$$

Il résulte de ce qui précède que $[x_p, x_{n+p}]$ sera infiniment petit avec $1/p$ indépendamment de n.

Inversement, démontrons que la suite des points x_n de F est convergente quand elle satisfait au critère de Cauchy généralisé. Dans ce but, prenons un point y_p de E assez voisin de x_p pour que, quel que soit p,

$$[y_p, x_p] < 1/p.$$

On aura

$$(y_p, y_{n+p}) = [y_p, y_{n+p}] \leq [y_p, x_p] + [x_p, x_{n+p}] + [x_{n+p}, y_{n+p}] < 2/p + [x_p, x_{n+p}];$$

le second membre est infiniment petit avec $1/p$, indépendamment de n. Donc

les distances (y_p, y_{n+p}) vérifient le critère de Cauchy: la suite des y_p est convergente. Soit y sa limite qui appartient à F. On aura

$$[x_p, y] \leq [x_p, y_p] + [y_p, y] < 1/p + [y_p, y].$$

Donc les x_p convergent (vers y).

XXXIX. *Dans un espace (D), tout ensemble compact et fermé possède la propriété de Borel-Lebesgue et réciproquement* (page **317**). J'ai démontré la réciproque pour l'espace (V) le plus général (**61**, page 15).

Le théorème direct a été démontré dans ma Thèse (**22**, page 26), moyennant une hypothèse restrictive dont on peut se débarasser suivant une remarque de M. Hildebrandt (**71**, page 353).

Mais cette démonstration repose sur le théorème analogue pour la propriété de Borel. Vue l'importance de la proposition énoncée ci-dessus, il est bon d'en donner une démonstration directe.

Supposons qu'une famille F d'ensembles I couvre un ensemble E. Tout point x de E étant intérieur à un ensemble de F, il y a au moins un nombre $\rho > 0$ tel que le sphéroïde de centre x, rayon ρ, appartienne entièrement à un même ensemble I de F. Soit ρ_x la borne supérieure de ρ pour un point x donné de E. (Si cette borne supérieure était infinie, tout E—et même tout l'espace—serait couvert par un seul des I). Soit S_x le sphéroïde de centre x, rayon $\rho_x/2$; il est *intérieur* à l'un au moins, I_x, des ensembles de F.

Pour démontrer le théorème, il suffit donc de démontrer qu'il existe un nombre fini de points x de E tels que tout point de E appartienne à l'un des sphéroïdes S_x, en nombre fini, correspondants.

Or, dans le cas contraire, soit a_1 un point quelconque de E; il existerait au moins un point a_2 de E n'appartenant pas à S_{a_1}. De même, il existerait au moins un point a_3 n'appartenant ni à S_{a_1}, ni à S_{a_2}. Si on a formé une suite de points $a_1, \cdots, a_n$ de E tels que a_n n'appartienne à aucun des ensembles $S_{a_1}, \cdots, S_{a_{n-1}}$, alors il existerait aussi un point a_{n+1} de E n'appartenant à aucun des ensembles $S_{a_1}, \cdots, S_{a_n}$. Ainsi le procédé devrait se continuer indéfiniment. Alors, on aurait une suite infinie T de points de E, $a_1, a_2, \cdots$ tous distincts et tels que, quel que soit n, a_n n'appartienne à aucun des ensembles $S_{a_1}, \cdots, S_{a_{n-1}}$.

Or puisque E est compact et fermé il y a au moins un point a de E qui est point d'accumulation de l'ensemble T. Il y a donc une infinité de points de T, distincts, aussi voisins que l'on veut de a. Prenons en particulier a_n et a_{p+n} à distances de a inférieures à $\rho_a/16$. Alors pour tout point x à distance de a_n au plus égale à $\rho_a/4$, on a

$$(x, a) \leq (x, a_n) + (a_n, a) < \rho_a/2.$$

Donc, tout point d'un sphéroïde de centre a_n, rayon $\rho_a/4$, appartient à S_a et,

par suite, à I_a. Il résulte alors de la définition de ρ_{a_n} que $\rho_{a_n} \geq \rho_a/4$, et comme a_{n+p} n'appartient pas à S_{a_n}, on aurait

$$(a_{n+p}, a_n) \geq \rho_{a_n}/2;$$

alors que

$$(a_{n+p}, a_n) \leq (a_{n+p}, a) + (a_n, a) < \rho_a/8.$$

On remarquera que cette démonstration est aussi simple que les démonstrations qui ont été données pour les espaces euclidiens.

7. *Ensembles parfaitement compacts.*

XL. *Dans un espace* (D), *il a identité entre la notion d'ensemble compact (en soi) et celle d'ensemble parfaitement compact (en soi)* (page **317**).

Une première démonstration s'obtient par voie indirecte en s'appuyant sur ce que, dans un espace (D), tout ensemble compact et fermé possède la propriété de Borel-Lebesgue et sur le théorème de Chittenden d'après lequel dans tout espace (V), tout ensemble possédant cette propriété est parfaitement compact en soi.

J'ai aussi donné de ce fait, énoncé par M. R. L. Moore, une démonstration directe (**71**, page 348) mais où j'adoptais pour la notion d'ensemble parfaitement compact la définition de cet auteur. Il faut alors compléter par la démonstration de l'équivalence de cette définition et de la définition adoptée dans mon livre. Cette équivalence a été établie par MM. Alexandroff et Chittenden.

On peut aussi donner une démonstration ne faisant intervenir que la définition des ensembles parfaitement compacts dérivée de celle due à MM. Sierpinski et Kuratowski, définition adoptée dans mon livre et utilisée ci-dessus.

Il s'agit de démontrer que si E est un ensemble compact et si e est un sous-ensemble infini de E, il existe un point a dont tout voisinage comprend à son intérieur une infinité de points de e ayant la puissance de e. Or s'il existait un tel point, il appartiendrait à la fermeture $F = E + E'$ de E. Cet ensemble est compact et fermé (**34**, page 4). Si E n'était pas parfaitement compact, alors pour tout point x de F, il y aurait un sphéroïde S_x de centre x ne comprenant à son intérieur qu'un ensemble de points de e de puissance inférieure à celle de e. Or F ayant la propriété de Borel-Lebesgue (voir XXXIX), l'ensemble F est couvert par un nombre fini des S_x. Donc l'ensemble e est compris dans la somme des intérieurs d'un nombre fini d'ensembles, chacun de ces intérieurs ne contenant qu'un ensemble de points de e d'une puissance inférieure à celle de e. D'où la contradiction annoncée.

XLI. Soit $U(x)$ une fonctionnelle uniformément continue sur un ensemble E compact et fermé. C'est-à-dire que pour tout nombre $\varepsilon > 0$, il existe un

nombre $\eta > 0$, tel que $| U(x) - U(x') | < \epsilon$ pour tout couple de points x, x' de E dont la distance est $< \eta$. On peut couvrir E au moyen de sphéroïdes de rayon η. Puisque E est compact et fermé, on peut couvrir E au moyen d'un nombre fini de ces sphéroïdes. On peut donc couvrir E par un nombre fini d'ensembles en l'intérieur de chacun desquels l'oscillation de $U(x)$ soit inférieure à un nombre positif arbitraire ϵ, donné d'avance.

Réciproquement, supposons cette condition vérifiée. Alors, on peut couvrir E au moyen d'un nombre fini d'ensembles $E_1, \cdots, E_n$ dans l'intérieur de chacun desquels l'oscillation de U soit $< \epsilon/3$, par exemple. Soient I_r l'intérieur de E_r, p_{rs} la borne inférieure des distances des points de $E \cdot I_r$ à ceux de $E \cdot I_s$ et η la plus petite de celles des bornes $\rho_{r,s}$ (en nombre fini) qui ne sont pas nulles.* Si deux points a, b, de E sont à distance $(a, b) < \eta$, ou bien ils appartiennent au même I_r et alors $| U(a) - U(b) | < \epsilon/3 < \epsilon$, ou bien ils appartiennent à I_r et I_s tels que $\rho_{r,s} = 0$. Dans ce cas, il y a une suite de couples a_p, b_p de points de $E \cdot I_r$ et $E \cdot I_s$ respectivement, tels que (a_p, b_p) tende vers zéro et, par suite, puisque E est compact et fermé, on peut extraire de cette suite, une suite de couples a_p', b_p' convergeant vers le même point c de E.

Ce point c appartient à un des I, soit I_m (distinct ou non de I_r et de I_s). Puisque c est intérieur à E_m, les points a_p', b_p' seront aussi intérieurs à E_m à partir d'un certain rang q. Alors on aura

$$(1) \qquad \begin{gathered} | U(a_p') - U(b_p') | < \epsilon/3, \quad | U(a) - U(a_p') | < \epsilon/3, \\ | U(b_p') - U(b) | < \epsilon/3 \end{gathered}$$

pour les trois couples de points appartenant respectivement à $E \cdot I_m$, $E \cdot I_r$, $E \cdot I_s$. D'où

$$(2) \qquad | U(a) - U(b) | < \epsilon.$$

Ainsi, *la condition nécessaire et suffisante pour qu'une fonctionnelle U, définie sur un ensemble E compact et fermé, y soit uniformément continue est que, pour tout nombre $\epsilon > 0$, il existe un nombre fini d'ensembles couvrant E et dans l'intérieur de chacun desquels l'oscillation de U est inférieure à ϵ* (page 279).

On pourrait se demander si l'on pourrait supprimer dans cet enoncé les mots "à l'intérieur de." On ne pourrait les supprimer purement et simplement, car alors toute fonctionnelle bornée serait uniformément continue, comme on le voit en considérant E comme la somme des ensembles E_α où $\alpha\epsilon \leq U(x) < (\alpha + 1)\epsilon$ et où α ne prend que des valeurs entières (> 0, < 0 ou $= 0$).

UNIVERSITÉ DE STRASBOURG,
December 29, 1926.

* Si tous les ρ_{rs} sont nuls, on prendra $\eta = 1$, par exemple.

On Certain Points in the Theory of Dirichlet Series.

By J. F. Ritt

The results of this paper, all of which deal with Dirichlet series, bear upon the following topics:

I. *Picard's Theorem.*
II. *Integral functions.*
III. *Asymptotic convergence.*

The results under each topic are given at the head of the section dealing with that topic.

I. Picard's Theorem.

1. *Statement of results.* We consider, in this section, a function $f(z)$, $(z = x + iy)$, analytic in a half-plane $x \leqq R$, and representable in a half-plane $x < T \leqq R$ by an absolutely convergent series

$$a_0 + a_1 e^{\lambda_1 z} + \cdots + a_n e^{\lambda_n z} + \cdots, \tag{1}$$

where

$$0 < \lambda_1 < \cdots < \lambda_n < \cdots; \quad \lim_{n\to\infty} \lambda_n = \infty.$$

We use positive coefficients for z, rather than the customary negative ones, in order to heighten the similarity between our results and the corresponding known results on functions defined by Taylor series. No knowledge of T is needed other than that of its existence.

The value a_0 will be attributed to $f(z)$ for $z = -\infty$.

Theorems I and II, below, generalize respectively the theorems of Landau and of Schottky. We call particular attention to the fact that, in those theorems, the λ's which follow λ_1, and the extent of the region of convergence of (1), do not figure. The proofs, based upon the use of a modular function, are very parallel to the non-elementary proofs of the Landau and Schottky theorems.* Theorems IV and III, lemmas to Theorem II, are generalizations of Schwarz's lemma, and of the theorem that the modulus of a function ana-

* More novelty of detail will probably be found in the proofs of Theorems III, VI, X, XI, XII than in the other proofs.

lytic in a circle assumes its maximum on the circumference. A theorem like Theorem III, but dealing only with exponential polynomials, is given by Bohr.*

THEOREM I. *Let $f(z)$, analytic for $x \leqq R$, be representable, for $x < T \leqq R$, by an absolutely convergent series* (1). *There exists a function $L(a_0, a_1, \lambda_1)$ such that, if $R > L(a_0, a_1, \lambda_1)$, then $f(z)$ takes on at least one of the values* 0 *or* 1 *in the half-plane $x \leqq R$.*

THEOREM II. *Let $f(z)$, analytic for $x \leqq R$, be representable, for $x < T \leqq R$, by an absolutely convergent series* (1). *Let θ be a variable assuming all values greater than zero. There exists a function $S(a_0, \lambda_1, \theta)$ such that, if $f(z)$ is nowhere zero or unity for $x \leqq R$, we have $| f(z) | < S$ for $x \leqq R - \theta$.*

It should be noticed that S is independent of R.

THEOREM III. *Let $f(z)$, analytic, bounded and non-constant for $x \leqq R$, have an absolutely convergent development* (1) *for $x < T \leqq R$. Let $M(x)$ represent, for $x \leqq R$, the least upper bound of $| f(z) |$ on the vertical line of abscissa x.*† *Then $M(x)$ is an increasing function of x for $x \leqq R$.*‡

THEOREM IV. *Let $f(z)$, analytic, with $| f(z) | \leqq h$, (h arbitrary), for $x \leqq R$, have, for $x < T \leqq R$, an absolutely convergent development*

$$f(z) = a_1 e^{\lambda_1 z} + \cdots + a_n e^{\lambda_n z} + \cdots,$$

with

$$0 < \lambda_1 < \cdots < \lambda_n < \cdots; \quad \lim_{n \to \infty} \lambda_n = \infty.$$

Then, for $x < R$,

$$f(z) \leqq h\, e^{\lambda_1 (x - R)},$$

equality being possible only in the case of $f(z) = a_1 e^{\lambda_1 z}$.

2. *Proof of Theorem I.* Suppose that $f(z)$ is nowhere zero or unity for $x \leqq R$. Then $a_0 \neq 0, 1$.

Let $\mu(z)$ be a function, with $\mu(0) = a_0$, which maps the circle $| z | < 1$ upon the entire plane with the points 0, 1, ∞ removed. Let $\nu(z)$ be the inverse of $\mu(z)$.

Taking that branch of $\nu(z)$ for which $\nu(a_0) = 0$, let us form the function $\nu[f(z)]$. As $f(z)$ assumes none of the values 0, 1, ∞, which are the only

* *Acta Mathematica*, Vol. 47 (1926), p. 245.

† $M(x)$ is the least upper bound of $f(x + iy)$ for x fixed and $-\infty < y < +\infty$.

‡ If $x_1 < x_2$, $M(x_1) < M(x_2)$.

singular points of $\nu(z)$, $\nu[f(z)]$ will be uniform, analytic and less in modulus than unity, for $x \leqq R$.

As $a_0 \neq 0, 1$, $\nu(z)$ is developable in a Taylor series in $z - a_0$. Because the development (1) of $f(z)$ is absolutely convergent, we can get a development for $\nu[f(z)]$, absolutely convergent in some half-plane, by substituting (1) for z, formally, in the expansion of $\nu(z)$ about a_0. We find thus, as $\nu(a_0) = 0$,

$$\nu[f(z)] = \nu'(a_0)a_1e^{\lambda_1 z} + \cdots. \tag{2}$$

The coefficients of z which follow λ_1 are linear combinations of the λ's in (1) with positive integral coefficients.

As the development in (2) has a half-plane of absolute convergence, and as $|\nu[f(z)]|$ is bounded for $x \leqq R$, we have the following expression for the coefficient of $e^{\lambda_1 z}$ in (2):*

$$\nu'(a_0)a_1 = \lim_{\omega \to \infty} (1/2\omega i) \int_{R-\omega i}^{R+\omega i} \nu[f(z)]e^{-\lambda_1 z}dz,$$

The integration is performed along the line $x = R$.

Thus, as $|\nu[f(z)]| < 1$, we have

$$|a_1\nu'(a_0)| < e^{-\lambda_1 R},$$

so that

$$R < -\{\log |a_1\nu'(a_0)|\}/\lambda_1.$$

This proves our theorem.

3. *Proofs of Theorems III and IV.* Let $f(z)$ satisfy the hypothesis of Theorem III.

A recent theorem of Doetsch † informs us that $\log M(x)$ is a downward convex function of x for $x \leqq R$. By this, it is meant that if the points $(x, \log M(x))$ are plotted in Cartesian coordinates, then, if $x_1 < x_2 < x_3$, the point $(x_2, \log M(x_2))$ does not lie above the line joining $(x_1, \log M(x_1))$ and $(x_3, \log M(x_3))$.

Our first step will be to show that, for x close to $-\infty$, $M(x) > |a_0|$. This is obvious when $a_0 = 0$. Suppose that $a_0 \neq 0$. We assume, without loss of generality, that $a_1 \neq 0$.

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* Valiron, "Théorie générale des séries de Dirichlet." *Mémorial des Sciences Mathématiques.* Paris, 1926, p. 15.

† *Mathematische Zeitschrift,* Vol. 8 (1920), p. 237.

We have for $x < T$,

$$f(z) = a_0 + a_1 e^{\lambda_1 z} + e^{\lambda_1 z}(a_2 e^{(\lambda_2-\lambda_1)z} + a_3 e^{(\lambda_3-\lambda_1)z} + \cdots).$$

As the series in parentheses converges absolutely for $x < T$, and therefore, because of the nature of Dirichlet series, uniformly for $x < T - \delta$, $(\delta > 0)$, we have

$$f(z) = a_0 + a_1 e^{\lambda_1 z} + \epsilon(z) e^{\lambda_1 z},$$

where, as x approaches $-\infty$, $\epsilon(z)$ approaches zero uniformly with respect to y.

Let x be so close to $-\infty$ that $|\epsilon(z)| < |a_1|/2$. For any such x, we can find a y such that $a_1 e^{\lambda_1 z}$ has the same argument as a_0. Then

$$|f(x+iy)| > |a_0 + a_1 e^{\lambda_1 x}| - |a_1 e^{\lambda_1 x}|/2,$$

so that $M(x) > |a_0|$.

We now prove that $M(x)$ is an increasing function. Suppose that $a_0 \neq 0$. We shall show first that $M(x) > |a_0|$ for every $x \leqq R$. Suppose that, for some x_1, $M(x_1) \leqq |a_0|$. Let x_2 and $x_3 > x_2$ be two numbers, less than x_1 and close to $-\infty$, for which $M(x_3) > M(x_2)$. Such points exist, because $M(x)$ approaches $|a_0|$ from above as $x \to -\infty$. Then $\log M(x_3)$ exceeds both $\log M(x_1)$ and $\log M(x_2)$, so that $\log M(x)$ cannot be convex downward. Thus $M(x_1) > |a_0|$.

Let now x_1 and $x_2 > x_1$ be any two numbers, and suppose that $M(x_1) \geqq M(x_2)$. For x_3 close to $-\infty$, $M(x_3)$, being close to $|a_0|$, is less than $M(x_1)$. We see again that $\log M(x)$ is not convex downward. Thus $M(x_1) < M(x_2)$, so that $M(x)$ is an increasing function.

For the case $a_0 = 0$, it is at once obvious, because $\log M(x)$ is convex downward and approaches $-\infty$ with x, that $M(x)$ is an increasing function. Theorem III is proved.

We take now Theorem IV. Suppose that $f(z)$ is not $a_1 e^{\lambda_1 z}$. Then $g(z) = e^{-\lambda_1 z} f(z)$ satisfies the hypothesis of Theorem III. When $x = R$, $|g(z)| \leqq h\, e^{-\lambda_1 R}$. Hence, for $x < R$, $|g(z)| < h\, e^{-\lambda_1 R}$, so that

$$e^{-\lambda_1 x}|f(z)| < h\, e^{-\lambda_1 R},$$

or

$$|f(z)| < h\, e^{\lambda_1 (x-R)}.$$

When $f(z) = a_1 e^{\lambda_1 z}$, if we take $h = |a_1|\, e^{\lambda_1 R}$, we get an equation, rather than an inequality.

4. *Proof of Theorem II.* Suppose that $f(z)$ is neither zero nor unity

for $x \leqq R$. Then $|\nu[f(z)]| < 1$ for $x \leqq R$. This, and (2), make $\nu[f(z)]$ satisfy the hypothesis of Theorem IV. Then

$$\nu[f(z)] \leqq e^{-\lambda_1 \theta}$$

for $x \leqq R - \theta$. As $f(z) = \mu\nu f(z)$, and as $\mu(z)$ is bounded for $|z| \leqq e^{-\lambda_1\theta} < 1$, we have an upper bound for $|f(z)|$ for $x \leqq R - \theta$, depending on a_0, λ_1 and θ. Theorem II is proved.

II. INTEGRAL FUNCTIONS.

5. *Statement of results.* We consider a non-constant function $f(z)$ defined by a series

$$a_0 + a_1 e^{\lambda_1 z} + \cdots + a_n e^{\lambda_n z} + \cdots, \tag{3}$$

with

$$0 < \lambda_1 < \cdots < \lambda_n < \cdots; \quad \lim_{n\to\infty} \lambda_n = \infty,$$

absolutely convergent for all values of z.

Of course, $f(z)$ is bounded in any left half-plane, and therefore along any vertical line. Let $M(x)$ be the least upper bound of $|f(z)|$ along a vertical line of abscissa x. We know from Theorem III that $M(x)$ increases with x. Then $M(x)$ must increase indefinitely, for $f(z)$ cannot be bounded in the entire plane. With respect to the growth of $M(x)$, we prove, below, the

THEOREM V. *If $M(x) \leqq e^{lx}$ $(l \geqq 0)$, for all sufficiently large values of x, then $f(z)$ is an exponential polynomial in which the greatest coefficient of z does not exceed l, that is, $a_n = 0$ for $\lambda_n > l$.*

We shall say that $f(z)$ is of order ρ if, for every $\epsilon > 0$, we have, for x sufficiently large,

$$M(x) < e^{e^{x(\rho+\epsilon)}}$$

and if, besides, there exists a sequence of values of x tending towards $+\infty$ for which

$$M(x) > e^{e^{x(\rho-\epsilon)}}$$

We shall say that a function $f(z)$, of order $\rho > 0$, belongs to the type σ of the order ρ if, for every $\epsilon > 0$,

$$M(x) < e^{(\sigma+\epsilon)e^{\rho x}},$$

whereas, for a sequence of x's tending towards $+\infty$,

$$M(x) > e^{(\sigma-\epsilon)e^{\rho x}}.$$

We shall study the influence of the coefficients in (3) on the rate of increase of $M(x)$, placing upon the λ's the quantitative restriction

$$\liminf_{n\to\infty} \lambda_n/\log n > 0. \tag{4}$$

This assumption, frequently made in investigations on Dirichlet series, holds, for instance, for ordinary Dirichlet series $\sum a_n n^z$.

The problem of determining whether $f(z)$ belongs to a certain order, differs in no material respect from the corresponding problem for integral functions defined by power series. This is due largely to the crudeness with which the order of a function measures its modulus. One has only to follow the power series investigation,* using (5), below, to establish the following result:

For $f(z)$, defined by the everywhere absolutely convergent series (3), for which (4) holds, to be of order ρ, it is necessary and sufficient that

$$\rho = -\limsup_{n\to\infty} (\lambda_n \log \lambda_n)/\log |a_n|.$$

On the other hand, the determination of conditions for $f(z)$ to belong to a certain type of a given order, presents essentially new features over the power series investigation. We prove, in this connection, the

THEOREM VI. *For $f(z)$, defined by the everywhere absolutely convergent series (3), with (4) holding, to belong to the type σ of the order ρ, where $\sigma > 0$, it is necessary and sufficient that*

$$\limsup_{n\to\infty} n^{1/\rho} \left(\sum_{\lambda_i = n}^{\lambda_i < n+1} |a_i| \right)^{1/n} = (\sigma e \rho)^{1/\rho}.$$

The summation is over all i's for which $n \leqq \lambda_i < n+1$.

For $\sigma = 0$, the condition is necessary, but it is sufficient only if $f(z)$ is not of order less than ρ.

6. *Proof of Theorem V.* Let $f(z)$ satisfy the hypothesis of Theorem V. We have, for any R,

$$a_n = \lim_{\omega\to\infty} (1/2\omega i) \int_{R-\omega i}^{R+\omega i} f(z)\, e^{-\lambda_n z}\, dz. \tag{5}$$

Thus, for x large,

$$|a_n| \leqq e^{(l-\lambda_n)x},$$

so that $a_n = 0$ if $\lambda_n > l$.

* Bieberbach, *Lehrbuch der Funktionentheorie*, Vol. 2, p. 228.

7. *Proof of Theorem VI.* Let $f(z)$ be defined by the everywhere absolutely convergent series (3) for which (4) holds. Let $\xi(z)$ represent the series

$$1 + e^{-\lambda_1 z} + \cdots + e^{-\lambda_n z} + \cdots,$$

which, because of (4), converges for $x > 0$. Let

$$g(z) = |a_0| + |a_1| e^{\lambda_1 z} + \cdots + |a_n| e^{\lambda_n z} + \cdots. \tag{6}$$

We shall prove that, for $h > 0$ and x arbitrary,

$$g(x-h) \leqq \xi(h)\, M(x),$$

where $M(x)$ is the least upper bound of $|f(z)|$ on the vertical line of abscissa x.

In short,* by (5), $|a_n| \leqq M(x)\, e^{-\lambda_n x}$. Thus, by (6),

$$g(x-h) \leqq M(x)\,(1 + e^{-\lambda_1 h} + \cdots) = \xi(h)\, M(x).$$

We shall show that *if $f(z)$ belongs to the type σ of the order ρ, then $g(z)$ belongs to the type σ of the order ρ.* Let $f(z)$ belong to the type σ of the order ρ. The least upper bound of $|g(z)|$ on a vertical line of abscissa x is $g(x)$. Now, for $\epsilon > 0$, $h > 0$, and for x large,

$$\begin{aligned} \log g(x) &\leqq \log \xi(h) + \log M(x+h) < \log \xi(h) + (\sigma+\epsilon) e^{\rho(x+h)} \\ &= \log \xi(h) + e^{\rho h}(\sigma+\epsilon) e^{\rho x}. \end{aligned}$$

An ϵ having been selected, let h be taken so small that

$$e^{\rho h}(\sigma+\epsilon) < \sigma + 2\epsilon.$$

Then

$$\log g(x) < \log \xi(h) + (\sigma + 2\epsilon) e^{\rho x}.$$

If, now, keeping ϵ and h fixed, we make x large, we have

$$\log g(x) < (\sigma + 3\epsilon) e^{\rho x},$$

so that $g(z)$ satisfies the first condition for belonging to the type σ of the order ρ. Furthermore, as $g(x) \geqq |f(z)|$, the second condition is also satisfied by $g(z)$.

Let

$$c_n = \sum_{\lambda_i = n}^{\lambda_i < n+1} |a_i|, \qquad (n = 0, 1, \cdots).$$

* We put $\lambda_0 = 0$.

Then, for $x > 0$,

$$g(x) \geqq c_0 + c_1 e^x + \cdots + c_n e^{nx} + \cdots,$$
$$g(x) < c_0 e^x + c_1 e^{2x} + \cdots + c_n e^{(n+1)x} + \cdots.$$

From the first inequality, we see that the integral function $\sum c_n z^n$ satisfies the first condition for belonging to the type σ of the order ρ in the ordinary sense of the theory of integral functions. The second inequality shows that $\sum c_n z^{n+1}$ satisfies the second condition. But if either power series satisfies one of the conditions, the other does also. Hence, to ask that $f(z)$ belong to the type σ of the order ρ, in the sense of this paper, is to ask that $\sum c_n z^n$ belong to the type σ of the order ρ in the ordinary sense. This gives our theorem immediately.*

III. ASYMPTOTIC CONVERGENCE.

8. *Statement of results.* We deal with series of the type

(7) $$a_0 + a_1 e^{-\lambda_1 z} + a_2 e^{-\lambda_2 z} + \cdots + a_n e^{-\lambda_n z} + \cdots,$$

with

$$0 < \lambda_1 < \cdots < \lambda_n < \cdots; \quad \lim_{n\to\infty} \lambda_n = \infty,$$

making no assumptions as to the convergence of the series.

Consider any domain containing points of arbitrarily large x, and a function $f(z)$ defined for all points of this domain whose x's exceed some value. We shall say that $f(z)$ is *asymptotically equivalent* to (7) in the given domain if, for every n,

$$f(z) = a_0 + a_1 e^{-\lambda_1 z} + \cdots + a_n e^{-\lambda_n z} + \epsilon_n(z)\, e^{-\lambda_n z},$$

where $\epsilon_n(z)$ approaches zero, uniformly with respect to y, as x increases indefinitely. We shall write, putting $\lambda_0 = 0$,

$$f(z) \sim \sum a_n e^{-\lambda_n z}.$$

For domains, we shall use generally, in this paper, sectors of the type $|\operatorname{amp} z| \leqq \phi \leqq \pi/2$. We shall call such a sector a *sector of opening* 2ϕ. It will be understood that $f(z)$ is analytic for all sufficiently distant points of the sector.

Our results follow.

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* Bieberbach, *loc. cit.*, p. 231.

THEOREM VII. *If the series* $\sum a_n e^{-\lambda_n z}$ *has a domain of convergence, then the function* $f(z)$ *represented by the series is asymptotically equivalent to the series in any sector of opening less than* π.

THEOREM VIII. *If* $f(z) \sim \sum a_n e^{-\lambda_n z}$ *and* $g(z) \sim \sum b_n e^{-\lambda_n z}$, *then* $f(z) + g(z)$ *and* $f(z)\, g(z)$ *are respectively equivalent, asymptotically, to the sum and to the Dirichlet product of the series for* $f(z)$ *and* $g(z)$.

It is unnecessary to specify a domain in Theorem VIII. The proofs of both items of that theorem are analogous to the proofs of the corresponding results for power series, and will be omitted. The result on multiplication has a special interest, in view of the fact that the product of two Dirichlet series may diverge at points for which both converge.

THEOREM IX. *If* $f(z) \sim \sum a_n e^{-\lambda_n z}$ *in a sector of opening* 2ϕ, *then* $f'(z) \sim \sum -\lambda_n a_n e^{-\lambda_n z}$ *in any sector of opening less than* 2ϕ.

Theorem IX is analogous to one which we proved several years ago for power series.* It will be seen that the proof really applies to more general domains than sectors.

THEOREM X. *Let*

$$\liminf \lambda_n / \log n > 0.$$

Let $f(z) \sim \sum a_n e^{-\lambda_n z}$ *in a sector of opening* π. *Then* $\sum a_n e^{-\lambda_n z}$ *converges absolutely to* $f(z)$ *in some half-plane.*

Theorem X, more hidden than those which precede, generalizes the theorem that a power series which is asymptotically equivalent to an analytic function for a complete neighborhood of ∞ converges for large values of the variable.

THEOREM XI. *If* $f(z)$ *and* $g(z)$ *are both equivalent asymptotically to* $\sum a_n e^{-\lambda_n z}$, *with unrestricted* λ's, *in a sector of opening* π, *then* $f(z) \equiv g(z)$.

THEOREM XII. (Existence theorem.) *Let* $\lambda_1, \lambda_2, \cdots$, *be any sequence of positive numbers, increasing towards* ∞, *and* $a_1, a_2, \cdots$, *be any numbers. The series*

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} (1 - e^{b_n z \log z}), \tag{8}$$

where the b's are so taken that, for every n, $b_n < 0$ *and*

$$| a_n b_n | < 2^{-n},$$

* *Bulletin of the American Mathematical Society*, Vol. 24 (1918), p. 225.

converges, for points of sufficiently large modulus, in any sector of opening less than π, to a function asymptotically equivalent, in that sector, to $\sum a_n e^{-\lambda_n z}$.

The coefficient of i in $\log z$ is understood to be taken between $-\pi$ and π.

The series (8) is similar to a series which we used some time ago for the construction of a function with any desired derivatives at a point.*

9. *Proof of Theorem VII.* The proof of Theorem VII is contained, in principle, in the proof of the familiar theorem that the series for $f(z)$ converges uniformly in any sector of opening less than π.†

No generality is lost in assuming that the series converges for $z = 0$. In fact, if the series converges for $z = R > 0$, we can deal with the function $f(z + R)$. The equivalence of $f(z + R)$ to $\sum a_n e^{-\lambda_n R} e^{-\lambda_n z}$ will imply that of $f(z)$ to $\sum a_n e^{-\lambda_n z}$. This is because the distant points of any sector of opening $2\phi < \pi$, with vertex at the origin, are contained within any sector of opening greater than 2ϕ with vertex at R.

We take any sector, with vertex at the origin, of opening $2\phi < \pi$.

Let

$$r_n(z) = \sum_{n+1}^{\infty} a_i e^{-\lambda_i z}, \quad A_n = \sum_{n+1}^{\infty} a_i.$$

Then, by partial summation,

$$r_n(z) = A_n e^{-\lambda_{n+1} z} + \sum_{n+1}^{\infty} A_i (e^{-\lambda_{i+1} z} - e^{-\lambda_i z}).$$

Now

$$\left| e^{-\lambda_{i+1} z} - e^{-\lambda_i z} \right| = \left| \int_{\lambda_i}^{\lambda_{i+1}} z\, e^{-uz}\, du \right| \leqq |z| \int_{\lambda_i}^{\lambda_{i+1}} e^{-ux}\, du$$
$$= (|z|/x)\,(e^{-\lambda_i x} - e^{-\lambda_{i+1} x}).$$

As the series for $f(z)$ converges for $z = 0$, there exists an $h > 0$ which exceeds every $|A_i|$. Then

$$|r_n(z)| < h\, e^{-\lambda_{n+1} x} + (h\,|z|/x) \sum_{n+1}^{\infty} (e^{-\lambda_i x} - e^{-\lambda_{i+1} x})$$
$$= h\,(1 + |z|/x)\, e^{-\lambda_{n+1} x}$$
$$< h\,(1 + \sec\phi)\, e^{-\lambda_{n+1} x}.$$

Because $\lambda_{n+1} > \lambda_n$, it is plain that $e^{\lambda_n z} r_n(z) \to 0$ as $x \to \infty$ in our sector. This proves the theorem.

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* *Annals of Mathematics*, Vol. 18 (1916), p. 18.

† Valiron, *loc. cit.*, p. 5.

10. *Proof of Theorem IX.* We have

$$f'(z) = -\lambda_1 a_1 e^{-\lambda_1 z} - \cdots - \lambda_n a_n e^{-\lambda_n z} + e^{-\lambda_n z}(-\lambda_n \epsilon_n(z) + \frac{d}{dz}\epsilon_n(z)).$$

Consider any ζ of large modulus in a sector of opening less than 2ϕ. The modulus of the derivative of $\epsilon_n(z)$ for $z = \zeta$, does not exceed the maximum modulus of $\epsilon_n(z)$ on the circle $|z - \zeta| = 1$. Hence the derivative of $\epsilon_n(z)$ approaches zero as $z \to \infty$ in the sector of opening less than 2ϕ. This proves our theorem.

5. *Proofs of Theorems X and XI.* Let $f(z)$ satisfy the hypothesis of Theorem X. There is an R such that $f(z)$ is analytic and bounded for $x \geqq R$. As we may replace $f(z)$ by $f(z + R)$, we assume that $R = 0$.

We shall prove, without using our assumption relative to the λ's, that, for every n,

$$a_n = \lim_{\omega \to \infty} (1/2\omega i) \int_{-\omega i}^{\omega i} f(z)\, e^{\lambda_n z} dz, \tag{9}$$

the integration being performed along the line $x = 0$.

As

$$f(z) = a_0 + a_1 e^{-\lambda_1 z} + \cdots + a_n e^{-\lambda_n z} + \epsilon_n(z)\, e^{-\lambda_n z},$$

where $\epsilon_n(z) \to 0$ as $x \to \infty$, we have

$$\int_{-\omega i}^{\omega i} f(z)\, e^{\lambda_n z} dz = \int_{-\omega i}^{\omega i} (a_0 + \cdots + a_{n-1} e^{-\lambda_{n-1} z})\, e^{\lambda_n z}\, dz + 2\omega i a_n + \int_{-\omega i}^{\omega i} \epsilon_n(z)\, dz. \tag{10}$$

The first integral in the second member of (10) stays bounded as $\omega \to \infty$. If, then, we can show that

$$\lim_{\omega \to \infty} (1/2\omega i) \int_{-\omega i}^{\omega i} \epsilon_n(z)\, dz = 0, \tag{11}$$

our formula for a_n will be established.

Let σ be any positive number. We have

$$\int_{-\omega i}^{\omega i} \epsilon_n(z) dz = \int_{-\omega i}^{\sigma + \omega i} \epsilon_n(z)\, dz + \int_{\sigma - \omega i}^{\sigma + \omega i} \epsilon_n(z)\, dz + \int_{\sigma + \omega i}^{\omega i} \epsilon_n(z) dz, \tag{12}$$

every path of integration being rectilinear.

Any $\eta > 0$ being taken, let a σ be chosen so that $|\epsilon_n(z)| < \eta$ for $x \geqq \sigma$.

Because $f(z)$ is bounded for $x \geqq 0$, $\epsilon_n(z) e^{-\lambda_n z}$ is bounded for $x \geqq 0$.

Hence $\epsilon_n(z)$ is bounded for $0 \leqq x \leqq \sigma$.* Let N be an upper bound for $|\epsilon_n(z)|$ in this region.

Then the first and third integrals in the second member of (12) have moduli not exceeding σN, whereas the modulus of the second integral is less than $2\omega\eta$. Hence

$$(13) \qquad \left| (1/2\omega i) \int_{-\omega i}^{\omega i} \epsilon_n(z)\, dz \right| < \sigma N/\omega + \eta.$$

If ω is large, the second member of (13) is less than 2η. This proves (11), and gives our formula for a_n.

Let

$$(14) \qquad r_n(z) = f(z) - a_0 - a_1 e^{\lambda_1 z} - \cdots - a_n e^{\lambda_n z}.$$

We shall prove that there exists an $S > 0$ such that $r_n(z) \to 0$ as $n \to \infty$, if $x > S$. This will prove that the asymptotic development converges to $f(z)$ in some half-plane. Furthermore, it is known that, for a Dirichlet series with $\liminf \lambda_n/\log n > 0$, the existence of a half-plane of convergence implies the existence of a half-plane of absolute convergence.† Thus, our theorem will be proved.

Let M be an upper bound for $|f(z)|$ on the imaginary axis. We see from (9) that $|a_n| \leqq M$. It follows from (14) that $|r_n(z)|$ has $(n+1)M$ for an upper bound on the imaginary axis.

The function $\epsilon_n(z) = e^{\lambda_n z} r_n(z)$ is bounded for $x \geqq 0$. We know from Doetsch's theorem that the least upper bound of $\log|\epsilon_n(z)|$, on a vertical line, is a downward convex function. Since $\log|\epsilon_n(z)| \to -\infty$, $x \to \infty$, the upper bound of $\log|\epsilon_n(z)|$ must decrease as x increases. Then the upper bound of $|\epsilon_n(z)|$ also decreases as x increases.

For $x = 0$, $|\epsilon_n(z)| = |r_n(z)|$, so that $|\epsilon_n(z)| \leqq (n+1)M$. Hence, for $x > 0$,

$$|r_n(z)| = e^{-\lambda_n x} |\epsilon_n(z)| < e^{-\lambda_n x} (n+1)M.$$

Our hypothesis with respect to λ_n implies the existence of a $\tau > 0$ such that $\lambda_n > \tau \log n$ for every n. Then, for every n,

$$|r_n(z)| < M(n+1)/n^{\tau x}.$$

It follows that, for $x > 1/\tau$, $r_n(z) \to 0$ as $n \to \infty$. This proves Theorem X.

We now prove Theorem XI. Because $(f(z) - g(z))e^{\lambda_n z}$ approaches zero as $x \to \infty$, the upper bound of the modulus of that function on a vertical line

* As well as for $x > \sigma$.

† Valiron, *loc. cit.*, p. 3.

decreases as x increases. The upper bound for $x = 0$ * is, for any λ_n, the upper bound of $f(z) - g(z)$. Then, if $x > 0$, $f(z) - g(z)$ must be zero, since $| e^{\lambda_n z} | \to \infty$ as $n \to \infty$.

12. *The existence theorem.* We consider the series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} (1 - e^{b_n z \log z}) \tag{15}$$

where, for every n, $b_n < 0$ and

$$| a_n b_n | < 2^{-n}. \tag{16}$$

We take any sector of opening $2\phi < \pi$, $(\phi > 0)$. For any z in this sector, let $z = r e^{i\theta}$, with $| \theta | < \phi$. Then

$$\begin{aligned} \mathcal{R}(z \log z) &= \mathcal{R}[(x + iy)(\log r + i\theta)] \\ &= x \log r - y\theta \\ &> x(\log r - \phi \tan \phi). \end{aligned} \tag{17}$$

Thus, when z is sufficiently large,

$$\mathcal{R}(z \log z) > 0. \tag{18}$$

Again, if z is a point of large modulus, in our sector,

$$| z \log z | < | z |^2 = x^2 \sec^2 \theta < x^2 \sec^2 \phi,$$

so that certainly, for $| z |$ large,

$$| z \log z | < e^x. \tag{19}$$

Let (18) and (19) both hold, in our sector, for $x > \sigma$. We say that (15) converges uniformly in the sector, for $x > \sigma$.

Let z be a point in the sector, with $x > \sigma$. Consider any n for which

$$| a_n | \leqq 2^{-n} | z \log z | . \tag{20}$$

We have, by (18) and (19),

$$\begin{aligned} | a_n e^{-\lambda_n z} (1 - e^{b_n z \log z}) | &< 2^{-n} | z \log z | \cdot e^{-\lambda_n x} (1 + 1) \\ &< 2^{-n+1} e^{-(\lambda_n - 1)x}. \end{aligned} \tag{21}$$

Consider any n for which (20) does not hold. We have

$$\begin{aligned} | 1 - e^{b_n z \log z} | &\leqq | b_n z \log z | + (1/2!) (| b_n z \log z |)^2 + \cdots \\ &< | b_n z \log z | e^{|b_n z \log z|}. \end{aligned} \tag{22}$$

* It is evidently legitimate to assume $f(z)$ and $g(z)$ analytic and bounded for $x \geqq 0$.

Using (16), and the fact that (20) does not hold, we have

$$| b_n z \log z | < 2^{-n} \left| \frac{z \log z}{a_n} \right| < 1,$$

so that, by (22),

$$| 1 - e^{b_n z \log z} | < | b_n z \log z | \cdot e. \tag{23}$$

Then, by (19) and (23),

$$| a_n e^{-\lambda_n z} (1 - e^{b_n z \log z}) | < e \cdot | a_n b_n | \cdot e^{-(\lambda_n - 1)x} < 2^{-n} e \cdot e^{-(\lambda_n - 1)x}. \tag{24}$$

From (21) and (24) we see that, for $x > \sigma$, in our sector, the terms of (15) are all less in modulus than the corresponding terms of

$$\sum_{n=1}^{\infty} 2^{-n} e \cdot e^{-(\lambda_n - 1)x}. \tag{25}$$

But (25) is evidently uniformly convergent for $x > 0$. Hence (15) is uniformly convergent, in our sector, for $x > \sigma$.

We show now that the function $f(z)$ to which (15) converges is asymptotically equivalent to $\sum a_n e^{-\lambda_n z}$.

It follows from (17) (last item), that as $x \to \infty$, with z in our sector,

$$\lim e^{\lambda z} \cdot e^{b_n z \log z} = 0 \tag{26}$$

for every $\lambda > 0$. We see from (25) that the remainder after n terms in (15) is of the form $\epsilon_n(z)\, e^{-(\lambda_n - 1)z}$, where $\epsilon_n(z) \to 0$ as $x \to \infty$.

Taking any n, let p be such that $\lambda_p > \lambda_n + 1$. Then

$$f(z) = \sum_{i=1}^{p} a_i e^{-\lambda_i z} (1 - e^{b_i z \log z}) + \epsilon_p(z) e^{-(\lambda_p - 1)z}.$$

For any i such that $n < i \leqq p$, we have

$$a_i e^{-\lambda_i z} (1 - e^{b_i z \log z}) = \eta_i(z)\, e^{-\lambda_n z},$$

where $\eta_i(z) \to 0$ as $x \to \infty$. Hence

$$f(z) = \sum_{i=1}^{n} a_i e^{-\lambda_i z} (1 - e^{b_i z \log z}) + \delta_n(z)\, e^{-\lambda_n z},$$

where $\delta_n(z) \to 0$ as $x \to \infty$. Then

$$f(z) = \sum_{i=1}^{n} a_i e^{-\lambda_i z} - e^{-\lambda_n z} \sum_{i=1}^{n} a_i e^{(\lambda_n - \lambda_i)z} \cdot e^{b_i z \log z} + \delta_n(z)\, e^{-\lambda_n z}$$

and this, by (26), proves that $f(z) \sim \sum a_n e^{-\lambda_n z}$ in our sector.

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On a Theorem of Severi.

By Oscar Zariski.

1. In one of his memoirs published in the *Mathematische Annalen*,* Professor Severi undertook to prove the following theorem:

Every correspondence between the points of the general curve of a linear system $|C|$, ∞^2 *at least, drawn on a regular surface, has a valence. In particular every plane curve varying in a linear system* ∞^2 *at least has only correspondences with valence* (*loc. cit.*, p. 516-517).

Severi gives two proofs of this theorem, one transcendental, based upon the investigation of the abelian integrals of the varying curve C of $|C|$, and the other purely geometric. But in each of them the theorem is proved only under certain conditions which the author points out explicitly. Thus, in the transcendental treatment Severi assumes that when a curve of the system acquires a new multiple point, this point will generally be a nodal double point, while in the geometric treatment of the question he makes the hypothesis that the system $|C|$, supposed to be ∞^r, should not have ∞^{r-1} reducible curves. Since the two conditions seem to him to be independent, he draws the conclusion that in the definite statement of the theorem both can be omitted and therefore the theorem is true without restrictions. Investigating closer the geometrical proof of Severi, certain doubts as to the rigor of the procedure were brought forth. There is a single weak point in Severi's reasoning. When this is eliminated, the necessary and sufficient conditions under which the result is true, can be completely formulated. The theorem thus limited still retains its importance, especially in that it is still able to prove the non-existence of singular correspondences on a curve of genus $p > 0$ with general modulus, which application was the first motive to Severi's investigation.

I formulate Severi's theorem thus:

I. *If in a net* $|C|$ *of degree* n *drawn on an algebraic regular surface the general curve* C *possesses a singular correspondence, the correspondence* $(n-1, n-1)$ *defined on* C *by the characteristic series* g_n^1 *is broken up into several irreducible correspondences, and at least one of them is singular.*

I also show:

* "Le corrispondenze singolari fra i punti di una curva variabile in un sistema lineare sopra una superficie regolare," *Mathematische Annalen*, Vol. 74 (1913).

II. *The above mentioned correspondences into which the correspondence* $(n-1, n-1)$ *is broken up form a base* * *on the general curve* C, *i. e. designating them by* $M_1, \cdots, M_k$, *for every other correspondence* M *on* C

$$\lambda M + \lambda_1 M_1 + \cdots + \lambda_k M_k \equiv 0,$$

where $\lambda, \lambda_1, \cdots, \lambda_k$ *are integers, and* $\lambda \neq 0$.

It may happen that the k correspondences M_i are not independent. Then the number of correspondences forming a base is reduced.

I further show that Severi's theorem is absolutely true for *simple* linear systems (consequently at least ∞^3), i. e., such that the curves of the system which pass through a given point P must not necessarily pass through other points varying with P. We have then the following theorem:

III. *On the general curve* C *of a simple linear system, drawn on a regular surface, all correspondences have a valence.*

2. Let $|C|$ be a linear system, ∞^2 at least, drawn on a regular surface F. and let us suppose that there exists on the general curve C a singular correspondence T, which can therefore vary continuously when C varies continuously in $|C|$. In general the correspondence T will not be determined rationally on every C, i. e. while C circulates in $|C|$, starting from an initial position and returning to it, T although preserving its indices will be transformed into another correspondence T', different from T. Severi shows that starting from T one can construct another correspondence also singular, which is rationally determined on every C. Possessing such a correspondence, which we shall call T, Severi reasons as follows: Let the curve C vary in a net H of degree n. Let x be any point on the surface F and let us consider the curve D_x described by all the points y which are homologous to x in the correspondences T relative to all the ∞^1 curves C passing through x. Let D_x have a multiplicity s at the point x and the multiplicities t—necessarily equal due to consideration of symmetry—at each of the $n-1$ points z in which the curves C passing through x intersect outside x and the base points of the net. D_x will also pass through the base points of the net with certain multiplicities which we can neglect because, when x varies on a fixed C, they give rise to fixed intersections of C with D_x. Varying x on a given C we obtain a continuous system ∞^1 of curves D_x, and since the surface is regular, the curves

* On the concept of a base, see F. Severi: "Sulle corrispondenze tra i punti di una curva algebrica e sopra certe classi di superficie," *Memorie della Reggia Accademia di Torino*, Vol. 64 (1903); "Sulla totalita delle curve algebriche tracciate sopra una superficie algebrica," *Mathematische Annalen*, Vol. 62 (1906).

D_x will be contained in one and the same linear system, by a well-known theorem of Enriques.* The curves D_x will consequently intersect C in equivalent groups of points. One such group, for a given D_x, consists of the point x, counted s times, of the group Y of the points homologous to x in the correspondence T on C, and of the group Z of the $n-1$ points z, each of which is counted t times. We have therefor

$$sx + Y + tZ \equiv sx' + Y' + tZ',$$

where x and x' are any two arbitrary points of C. But since $x + Z \equiv x' + Z'$, we deduce

$$Y + (s-t)x \equiv Y' + (s-t)x',$$

which proves that T has a valence $s - t$, which contradicts the hypothesis. The objection which may be raised against Severi's reasoning is that it is not necessarily true that D_x will pass through all the $n-1$ points z of the group Z with the same multiplicity. It would have been so, if, when x circulates on C starting out from an initial position and returning to it, any point z could be transformed into any one of the remaining $n-2$ points of the group Z, in other words, when the correspondence $(n-1, n-1)$ associating x and the points of the group Z is irreducible. But if this correspondence breaks up into several correspondences, and if we call $z_1, z_2, \cdots, z_r$ $(r < n-1)$ the points homologous to x in one of these correspondences, all we can say is that D_x will pass with the same multiplicity through the points $z_1, z_2, \cdots, z_r$, but the multiplicities relative to two different correspondences may well be different.

Suppose that the correspondence $(n-1, n-1)$ is broken up into k correspondences $M_1, \cdots, M_k$ and let $Z_1, \cdots, Z_k$ respectively be the groups of points homologous to x in the correspondences $M_1, \cdots, M_k$. If we denote by t_i the multiplicity with which the curve D_x passes through the points of the group Z_i, we will have instead of Severi's relation the following:

$$sx + Y + t_1Z_1 + \cdots + t_kZ_k \equiv sx' + Y' + t_1Z_1' + \cdots + t_kZ_k'.$$

We conclude that the correspondence

$$sI + T + t_1M_1 + \cdots + t_kM_k,$$

where I denotes the identical correspondence, has a valence, in symbols

$$(1) \qquad sI + T + t_1M_1 + \cdots + t_kM_k \equiv 0.$$

* F. Enriques, "Una proprietà delle serie continue di curve appartenenti ad una superficie algebrica regolare," *Rendiconti del Circolo Matematico di Palermo*, Vol. 13 (1899).

From this we conclude that if all the correspondences T_i $(i = 1, 2, \cdots, k)$ have a valence, then T has a valence, and this proves our theorem I (§ 1). It is easily shown that the correspondences $M_1, \cdots, M_k$, or a certain number of them in the case where they are not independent, form a base, i. e. for any other correspondence M on C we have

$$\lambda M + \lambda_1 M_1 + \cdots + \lambda_k M_k \equiv 0, \tag{2}$$

where $\lambda, \lambda_1, \cdots, \lambda_k$ are integers and $\lambda \neq 0$. In fact, if M is rationally determined on C then (2) is satisfied for $\lambda = 1$, in virtue of (1).* If, however, M is not rationally determined, we can construct, by Severi's method, another correspondence M', which is rationally determined on every C. The correspondence M', constructed thus, is related to M by $M' \equiv \mu M$, where μ is a positive integer. Since

$$M' + \lambda_1 M_1 + \cdots + \lambda_k M_k \equiv 0$$

we have also

$$\mu M + \lambda_1 M_1 + \cdots + \lambda_k M_k \equiv 0,$$

which proves the theorem II (§ 1).

3. In fact, one can give examples of nets of curves on regular surfaces, for instance, on the plane, in which the general curve possesses singular correspondences. There exist in the plane nets of harmonics or equianharmonic cubics,† and every such cubic possesses singular correspondences. Such nets of cubics have been studied recently by Miss Rossi (Rome) in her thesis. Miss Rossi finds an interesting class of nets of equianharmonic cubics with six base points, for which the characteristic series g_3^1 is cyclic, i. e. is generated by a birational cyclic transformation of third order. This is in accordance with our general theorem I, since in this case the characteristic series gives rise to a correspondence (2, 2) which, as it has to be reducible, breaks up necessarily into two singular birational transformations, one the inverse of the other. Note that these nets are generally deprived of a pencil of reducible curves.

In the examples mentioned above, the curves of the net have the same modulus. We can show that this is general for nets of elliptical curves, i. e. *if in a net of elliptical curves the general curve possesses a singular corre-*

* Note that the correspondence I can be eliminated from (1), since evidently $I + M_1 + M_2 + \cdots + M_k \equiv 0$.

† See O. Chisini, "Sui fasci di cubiche a modulo costante," *Rendiconti del Circolo Matematico di Palermo*, Vol. 40 (1916).

spondence, the curves of the net have the same modulus. This is seen by considering the transcendental representation of a singular correspondence T on an elliptical curve C. Let $y^{(1)}, y^{(2)}, \cdots, y^{(\beta)}$ be the homologous points of x in T. Then we have

$$\sum_1^\beta{}_\nu\, u(y^\nu) \equiv \pi u(x) + \pi_1 (\text{mod } 1, \omega),$$

where u is an abelian integral of the first kind, the periods of which are 1 and ω, and π, π_1 are constants. There exist also a known relation between π and ω:

$$\pi = h + g\omega, \quad \pi\omega = H + G\omega,$$

where h, g, H, G are integers and $g \neq 0$, if T is singular. The intergers h, g, H, G are called by Severi the characteristic integers of the correspondence. The period ω satisfies an equation of second degree

$$g\omega^2 + (h - G)\omega - H = 0. \tag{3}$$

If the curve C varies in a net in such a way that also the correspondence T varies continuously, the characteristic integers of T must remain unaltered, since they cannot vary continuously. Equation (3) shows that the period ω, which is also the modulus of C, does not vary with C. Q. E. D.

4. We will show now that Severi's theorem is applicable without restrictions for simple linear systems (consequently at least ∞^3). We have only to make a preliminary observation. Let us have on an irreducible curve C a linear involution g_n^1. We will designate by G the monodromic group of the g_n^1, i. e. the monodromic group of the n-valued algebraic function $y(x)$, which we obtain cutting the g_n^1 on C by the lines of the pencil $x = \text{const}$. It is known that G is transitive when C is irreducible. Let us consider the correspondence $(n-1, n-1)$ defined by the g_n^1. If this correspondence is broken up into several correspondences, what can we say about the group G? It is evident that in this case the substitutions of G which leave one letter unaltered, do not permute transitively the remaining $n-1$ letters, i. e. the group G is only simply transitive. If, on the other hand, the correspondence $(n-1, n-1)$ is irreducible, the group G is at least doubly transitive. In particular, when the group G is the symmetrical group on n letters, then the correspondence $(n-1, n-1)$ is certainly irreducible.

Let us take now on a regular surface F a linear system $|C|$, at least ∞^3 and simple. With this hypothesis we are certain that taking in $|C|$ a general net, the characteristic series g_n^1 of a general C of the net is simple,

i. e. its groups are not composed of the sets of points of an involution of lower order, and therefore its monodromic group G will be primitive. On the other hand, if two curves of $|C|$ are tangent to each other at a point P, they will have a simple contact there, and will not have another point of tangency. Therefore the monodromic group G of the above mentioned characteristic series g_n^1 will contain a transposition. The group being primitive, we deduce by means of a well known theorem in the theory of groups, that it will be the symmetrical group on n letters. That is sufficient to affirm that the correspondence $(n-1,\ n-1)$ on the general curve taken in a general net of $|C|$ is irreducible, and hence, by the general theorem I, C does not contain singular correspondences.

The application of this result to the proof of the non-existence of singular correspondences on a curve of genus p with general modulus is immediate. It is sufficient to show that there exist particular curves of genus p, which do not contain singular correspondences. This follows simply from the fact that for every value of p there exist simple linear systems of plane curves of genus p. In fact, given p, we determine the integer n such that

$$(n-1)(n-2)/2 < p \leq n(n-1)/2.$$

The curves of order $n+1$ with $n(n-1)/2 - p \leqq n-2$ fixed double points form evidently such a linear system.

In fact, the system contains degenerate curves ϕ_{n+1} broken up in $d = n(n-1)/2 - p$ lines forming a polygon whose vertices are at the d fixed double points and in $n+1-d \geqq 3$ arbitrary lines, hence the system is simple. The above mentioned degenerate curves ϕ_{n+1} relative to different polygons do not possess a common part and other common double points outside the d fixed points. Moreover the curves of the system are not composed of the curves of a pencil, due to the fact that these degenerate curves contain $n+1-d$ arbitrary lines. Hence, by the theorem of Bertini, the system is irreducible and of genus equal precisely to p.

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On Gierster's Classnumber Relations.

By J. V. Uspensky.

In the fifth of the papers published under the common title: "Sur les relations entre les nombres des classes des formes quadratiques binaires et positives" * I gave an elementary and purely arithmetical proof of Gierster's classnumber relations. The exposition could have been greatly simplified had I known at that time the explicit expressions for certain numerical functions. Lacking these expressions I had to resort to indirect means to eliminate the numerical function whose expression was not known *à priori*, which made my method unnecessarily complicated. Recently I discovered the lacking expressions, which are by no means obvious, and this makes it possible for me now to present my proof in its simplest form. My method is based on certain very general identities which may be useful on many occasions, and I deem it advisable to reproduce those identities with their proof, inasmuch as the proof is very simple and does not require much space.

1. *The Fundamental Identities.* Let $F(x, y, z)$ be an arbitrary function defined for integral values of its arguments and satisfying the following conditions of parity:

$$F(-x, y, z) = F(x, y, z)\,;\; F(x, -y, -z) = -F(x, y, z)\,,$$

and we suppose

$$F(x, 0, z) = 0$$

whenever the middle argument y can assume the value 0. On the other hand, n being an arbitrary positive integer, we consider two modes of partition of this number, the first one being indicated by the equation

$$n = i^2 + 2d\delta \tag{1}$$

where i may have any integral value, while d and δ are supposed to be positive integers. The second mode of partition is indicated by the equation

$$n = h^2 + \Delta\Delta' \tag{2}$$

where again h may be any integer, while Δ and Δ' are supposed to be positive integers, satisfying the additional conditions

$$\Delta < \Delta', \quad \Delta' \equiv \Delta \pmod 2.$$

* *Bulletin de l'Académie des Sciences de l'USSR*, No. 8-9, 1926.

This being so, let us consider the sum

$$S = \sum \{F[(\Delta' + \Delta)/2, (\Delta' - \Delta)/2, \Delta - h] - F[h, (\Delta' - \Delta)/2, \Delta - h]\}$$

extended over all the partitions (2). This sum may be naturally split into three parts S_1, S_2, S_3 according to the positive, negative, or zero value of $\Delta' - \Delta + 2h$. It is easy to see that $S_3 = 0$ except when n is a perfect square $= s^2$ in which case

$$S_3 = \sum \{F(s, j, s) - F(j, j, s)\}, \quad (j = 1, 2, 3, \cdots, s-1).$$

As to the part S_2, it vanishes in every case. For, supposing that h, Δ, Δ' run over the set of partitions (2) for which $\Delta' - \Delta + 2h < 0$ it is obvious that the numbers defined by

$$h_1 = -(\Delta' + \Delta)/2, \quad \Delta_1 = (\Delta - \Delta')/2 - h, \quad \Delta_1' = (\Delta' - \Delta)/2 - h$$

reproduce exactly the same set. We have therefore

$$S_2 = \sum \{F[(\Delta_1' + \Delta_1)/2, (\Delta_1' - \Delta_1)/2, \Delta_1 - h_1] - F[h_1, (\Delta_1' - \Delta_1)/2, \Delta_1 - h_1]\}.$$

But as

$$F[(\Delta_1' + \Delta_1)/2, (\Delta_1' - \Delta_1)/2, \Delta_1 - h_1] = F[-h, (\Delta' - \Delta)/2, \Delta - h] = F[h, (\Delta' - \Delta)/2, \Delta - h],$$

$$F[h_1, (\Delta_1' - \Delta_1)/2, \Delta_1 - h_1] = F[-(\Delta' + \Delta)/2, (\Delta' - \Delta)/2, \Delta - h) = F[(\Delta' + \Delta)/2, (\Delta' - \Delta)/2, \Delta - h],$$

we get the equation

$$S_2 = -S_2; \quad \text{that is } S_2 = 0.$$

It remains to consider the part S_1 extended over the partitions satisfying the inequality $\Delta' - \Delta + 2h > 0$. Suppose we let h, Δ, Δ' run over all the partitions of this kind. It is easy to see that the numbers i, d, δ defined either by

$$i = \Delta - h, \quad d = \Delta, \quad \delta = (\Delta' - \Delta + 2h)/2$$

or by

$$i = h - \Delta, \quad d = (\Delta' - \Delta + 2h)/2, \quad \delta = \Delta$$

reproduce all the solutions of the equation (1) satisfying either the condition $\delta + i - d > 0$ or the condition $\delta + i - d < 0$. As in the first case we have

$$F(\delta + i, \delta + i - d, i) = F[(\Delta' + \Delta)/2, (\Delta' - \Delta)/2, \Delta - h],$$

and in the second

$$F(\delta + i, \delta + i - d, i) = -F[h, (\Delta' - \Delta)/2, \Delta - h],$$

it follows from the preceding remarks

$$S_1 = \sum F(\delta + i,\ \delta + i - d,\ i)$$

the sum being extended over all the partitions (1). The three parts of the sum S being thus transformed, we finally get the following important identity:

$$\text{(I)} \qquad \sum F(\delta + i,\ \delta + i - d,\ i) = \sum \{F[(\Delta' + \Delta)/2,\ (\Delta' - \Delta)/2,\ \Delta - h] - F[h,\ (\Delta' - \Delta)/2,\ \Delta - h]\} + T,$$

where the sums are extended over the partitions (1) and (2) respectively and

$$T = \{\sum (F(j, j, s) - F(s, j, s))\},$$
$$(j = 1, 2, 3, \cdots, s-1,\ n = s^2,\ s > 0),$$

using the same abbreviated notation as in our paper: "On Jacobi's arithmetical theorems concerning the simultaneous representation of numbers by two different quadratic forms." * Repeating word by word the same reasoning, we can establish the second fundamental identity involving the function which satisfies the conditions:

$$F(-x, y, z) = -F(x, y, z);\ F(x, -y, -z) = F(x, y, z)$$
$$F(x, 0, z) = 0.$$

This second identity is

$$\text{(II)} \quad \sum F(\delta + i,\ \delta + i - d,\ i) = \sum \{F[(\Delta' + \Delta)/2,\ (\Delta' - \Delta)/2,\ \Delta - h] + F[h,\ (\Delta' - \Delta)/2,\ \Delta - h]\} + T,$$

where

$$T = \{\sum (F(j, j, s) - F(s, j, s)),$$
$$(j = 1, 2, 3, \cdots, s-1,\ n = s^2,\ s > 0),$$

the extent of summations being the same.

The identities (I) and (II) constitute the essential foundation of what follows. There are other important identities of similar nature which we hope to publish in connection with our investigations on incomplete numerical functions.

2. *Important Relation derived from* (I) *and* (II). *Let* $f(x)$ denote an absolutely arbitrary *even* function. We can take in the identities (I) and (II)

$$F(x, y, z) = (-1)^{z/2}[(-1)^x + (-1)^y] z f(x),$$
$$F(x, y, z) = (-1)^{z/2}[(-1)^x + (-1)^y] x f(x),$$

* *Transactions of the American Mathematical Society*, Vol. 30 (1928).

respectively, provided y is different from 0; but in both cases

$$F(x, 0, z) = 0.$$

Substituting furthermore $4n$ for n after a simple discussion we reach two following equations.

$$\sum_{(a)} (-1)^{i+\delta}(\delta + 2i)f(\delta + 2i) = \sum_{(b)} (-1)^{h+\Delta'}(\Delta' + \Delta)f(\Delta' + \Delta) -$$
$$- \sum_{(c)} (-1)^{(\Delta-1)/2+(h-1)/2} hf(h) + 2\sum_{(d)} (-1)^{j+k} jf(2j) + T,$$
$$\sum_{(a)} (-1)^{i+\delta}(\delta + 2i) = \sum_{(b)} (-1)^{h+\Delta'} 2\Delta f(\Delta' + \Delta)$$
$$- \sum_{(b)} (-1)^{h}\Delta[(-1)^{\Delta} + (-1)^{\Delta'}]f(2h) -$$
$$- \sum_{(c)} (-1)^{(\Delta-1)/2+(h-1)/2} hf(h) + 2\sum_{(d)} (-1)^{j+k}(j-k)f(2j) + T'.$$

The sums bearing the signs (a), (b), (c), (d) are extended over the partitions indicated below by the same signs, namely

$$(a)\ n = i^2 + d\delta; \quad (b)\ n = h^2 + \Delta\Delta',\ \Delta < \Delta';$$
$$(c)\ 4n = h^2 + \Delta\Delta,\ \Delta < \Delta',\ h \text{ odd}; \quad (d)\ n = j^2 + k^2,\ j > 0,\ k > 0,$$

while T and T' are both zeros except when n is a perfect square: $n = s^2$, $s > 0$, in which case

$$T = \Sigma\,[(-1)^{s+j} jf(j) - (-1)^{s}(1 + (-1)^{j}) sf(2s)],$$
$$T' = \Sigma\,[(-1)^{s+j} 2sf(j) - (-1)^{s}(1 + (-1)^{j}) sf(2s)],$$

the sums being extended over $j = 1, 2, 3, \cdots, 2s - 1$. Subtracting now the second equation from the first we come to an important relation

$$\text{(III)} \quad \sum_{(b)} (-1)^{h+\Delta'}(\Delta' - \Delta)[f(\Delta' + \Delta) - f(2h)]$$
$$= \sum_{(a)} (-1)^{i+d} d[f(d + 2i) - f(2i)] + U,$$

the extent of summations being as above and with the usual symbolism

$$U = \{\sum_{j=1}^{2s-1} (-1)^{s+j}(2s - j)f(j) + (-1)^{s} sf(0)\},\ n = s^2,\ s > 0.$$

The preceding equation can be exhibited in a different form. Denote as usual by $G(n)$ the classnumber of all quadratic forms corresponding to a given determinant $-n$, this number being diminished by $\frac{1}{2}$ when n is a perfect square, and by $\frac{2}{3}$ when n is a triple of a perfect square; moreover $G(0) = -\frac{1}{12}$. The notation $F(n)$ will be used to denote the classnumber of

those forms which can represent odd numbers, this number being diminished by ½ when n is a square of an *odd* number, while $F(0) = 0$. Putting

$$T(n) = 3G(n) - 2T(n), \quad T(0) = -\tfrac{1}{4}$$

it follows from the well-known Kronecker's classnumber relations

$$(-1)^m \sum T(m - k^2) = 2 \sum (-1)^{\Delta'}(\Delta' - \Delta), \quad m = \Delta\Delta', \quad \Delta < \Delta'$$

the first sum being extended over *all* integers k whose squares *do not exceed* an arbitrary positive integer m. By means of the last equation it is possible to present the equation (III) in the following final form: put for brevity

$$\psi(m) = \sum (-1)^{d'}(d' - d) f(d' + d) - \tfrac{1}{2}(-1)^m \sum T(m - h^2) f(2h),$$

the first sum being extended over all conjugate divisors of m satisfying the condition $d < d'$ while the second is extended over all integers h whose squares *do not exceed* m; with this notation adopted we have

$$\text{(IV)} \quad \sum (-1)^i \psi(m - i^2) = \sum (-1)^{i+d} d[f(d + 2i) - f(2i)] + W,$$

the first sum being extended over all integers i satisfying the inequality $i^2 < n$ while the second sum is extended over all the solutions of the indeterminate equation

$$n = i^2 + d\delta$$

with positive integers d and δ. As to W we have

$W = 0$, when n is not a perfect square,

$$W = (-1) \sum_{k=1}^{2s-1} (-1)^k (2s - k) f(k) + (-1)^s s f(0) + [(-1)^s/4]\,[f(2s) - f(0)]$$

when $n = s^2$, $s > 0$.

For particular choice of an arbitrary even function $f(x)$ the equation (IV) gives means to obtain different classnumber relations. In particular by taking $f(x) = \cos(2\pi x/3)$ or $f(x) = \cos(2\pi x/5)$ we obtain Gierster's classnumber relations of the third and fifth degrees (dritter und fünfter Stufe) respectively.

3. *The Equation derived from* (IV) *when* $f(x) = \cos(2\pi x/3)$. By taking $f(x) = \cos(2\pi x/3)$ in the fundamental equation (IV) we obtain, as a consequence of this assumption, an equation from which Gierster's relations of the third degree, and even more general, can be easily derived. To represent the right-hand member of (IV) in a suitable form it should be remembered that the number of representations of any integer n by the form

$$x^2 + y^2 + 3z^2 + 3t^2$$

is given by the sum

$$(-1)^n 1\tfrac{6}{3} \sum (-1)^d d \sin^2 (2\pi d/3)$$

extended over all the divisions of n. Hence, it is easy to see that the right-hand member of (IV) may be represented thus:

$$-\tfrac{3}{8} \sum (-1)^h \sum (-1)^{i+j+k+l} \cos(2\pi i/3)$$

where the inner sum is extended over all solutions of the equation

$$n - h^2 = i^2 + j^2 + 3k^2 + 3l^2$$

while h ranges over all integers whose squares are *less* than n. From the equation

$$\sum (-1)^h \psi(n-h^2) = -\tfrac{3}{8} \sum (-1)^h \sum (-1)^{i+j+k+l} \cos(2\pi i/3)$$

holding true for *every* n it follows necessarily

$$\psi(n) = -\tfrac{3}{8}(-1)^n \sum \cos(2\pi i/3), \quad n = i^2 + j^2 + 3k^2 + 3l^2,$$

that is

$$\sum T(n-h^2) \cos(2\pi h/3)$$
$$= (-1)^n 2 \sum (-1)^{d'}(d'-d) \cos 2\pi(d'+d)/3 + \tfrac{3}{4} \sum \cos(2\pi i/3).$$

On the other hand, we have

$$\sum T(n-h^2) = 2(-1)^n \sum (-1)^{d'}(d'-d), \quad n = dd', \; d < d'$$

whence it follows

$$\sum [3G(n-9h^2) - 2F(n-9h^2)] = 2(-1)^n \sum (-1)^{d'}(d'-d) + \tfrac{1}{2} \sum \cos(2\pi i/3), \tag{3}$$

the first sum in the right-hand member being extended now over conjugate divisors of n which satisfy the conditions:

$$d < d', \quad d' + d \equiv 0 \pmod 3. \tag{4}$$

To the equation (3) we can adjoin another equation

$$3 \sum [2F(n-9h^2) - G(n-9h^2)] = \tfrac{1}{4} N(n = 9x^2 + y^2 + z^2 + t^2) \tag{5}$$

as immediate consequence of Gauss' theorem concerning the number of representations by the sum of three squares. Now from (3) and (5) we can derive the expression for both sums

$$\sum F(n-9h^2), \quad \sum G(n-9h^2),$$

provided we can express their right-hand members in terms of the known numerical functions. But this makes it necessary to consider separately the three cases:

$$n \equiv 0, \quad n \equiv 2, \quad n \equiv 1 \pmod 3.$$

4. *Discussion of the case* $n \equiv 0 \pmod 3$. In this case the condition $d' + d \equiv 0 \pmod 3$ requires that d' and d should be separately divisible by 3 so that we have to evaluate the sum

$$\sigma = (-1)^n 6 \sum (-1)^{d'}(d' - d), \quad n = 9dd', \; d < d'.$$

Using the notations adopted by Kronecker

$$\Phi(n) = \sum d, \qquad n = dd',$$
$$\Psi(n) = \sum (d' - d), \quad n = dd', \; d < d',$$

we find easily: for an *odd* n

$$\sigma = 6\,\Psi(n/9)$$

and for an *even* n

$$\sigma = 12\Psi(n/36) - 12\Phi(n/36) + 6 \sum (-1)^d d, \quad n = 9dd'.$$

On the other hand, we obviously have

$$\sum \cos(2\pi i/3) = N(n = i^2 + j^2 + 3k^2 + 3l^2);$$

and it is easy to verify that

$$N(n = 9x^2 + y^2 + z^2 + t^2) = \tfrac{1}{4}N(n = x^2 + y^2 + z^2 + t^2) + \\ + \tfrac{3}{4}N(n/9) = x^2 + y^2 + z^2 + t^2).$$

Thus we see that the right-hand members of (3) and (5) may be expressed by means of the known numerical function and there is no further difficulty in obtaining the following final expressions: for an *odd* n

$$(6) \qquad \sum F(n - 9h^2) = \tfrac{1}{2}\Phi(n) - \Phi(n/3) + \tfrac{3}{2}\Psi(n/9),$$

$$(7) \qquad \sum G(n - 9h^2) = \Phi(n) - \tfrac{8}{3}\Phi(n/3) + 3\Psi(n/9),$$

and for an *even* n

$$(8) \qquad \sum F(n - 9h^2) = \mathrm{X}(n/3) - 3\Phi(n/36) + 3\Psi(n/36),$$

$$(9) \qquad \sum G(n - 9h^2) = 2\Phi(n/6) - 6\Phi(n/36) + 6\Psi(n/36),$$

where $\mathrm{X}(n)$ is defined by

$$\mathrm{X}(n) = \sum d, \quad n = d\delta, \quad \delta \text{ odd}.$$

5. *Discussion of the case* $n \equiv 2 \pmod 3$. In this case the condition $d' + d \equiv 0 \pmod 3$ is always verified and we obtain for the sum

$$\sigma = 2(-1)^n \sum (-1)^{d'}(d' - d),\ n = dd',\ d < d',\ d' + d \equiv 0 \pmod 3$$

the following expressions: for an *odd* n

$$\sigma = 2\Psi(n)$$

and for an *even* n

$$\sigma = 4\Phi(n/4) - 4\Phi(n/4) + 2 \sum (-1)^d d,\quad n = dd'.$$

Furthermore

$$\sum \cos(2\pi i/3) = -\tfrac{1}{2}N\ (n = i^2 + j^2 + 3k^2 + 3l^2),$$

and

$$N\ (n = 9x^2 + y^2 + z^2 + t^2) = \tfrac{1}{2}N\ (n = x^2 + y^2 + z^2 + t^2).$$

It is easy now to arrive at the following final expressions: for an *odd* n

$$\sum F(n - 9h^2) = \tfrac{1}{2}\Psi(n), \tag{10}$$

$$\sum G(n - 9h^2) = \Psi(n) - \tfrac{1}{3}\Phi(n); \tag{11}$$

and for an *even* n

$$\sum F(n - 9h^2) = \Psi(n/4) - \Phi(n/4) + \tfrac{1}{2}\mathrm{X}(n), \tag{12}$$

$$\sum G(n - 9h^2) = 2\Psi(n/4) - \Phi(n/4) + \tfrac{1}{2}\mathrm{X}(n). \tag{13}$$

6. *Discussion of the case* $n \equiv 1 \pmod 3$. The condition $d' + d \equiv 0 \pmod 3$ is never satisfied in this case, so that we have $\sigma = 0$ using the same notation as above. Moreover

$$\sum \cos(2\pi i/3) = \tfrac{1}{4}N(n = i^2 + j^2 + 3k^2 + 3l^2)$$

but the expression for the number of solutions of the equation

$$n = 9x^2 + y^2 + z^2 + t^2$$

is not so obvious in this case. However, there is no great difficulty in finding this number when n is *even*. For if n be even both the equations

$$4n = 9x^2 + y^2 + z^2 + t^2,\quad n = 9x^2 + y^2 + z^2 + t^2$$

have exactly the same number of solutions, and it suffices to consider the first of them. Suppose we consider the totality of repreesntations of $4n$ by the sum of 4 squares. Let P_1 denote the number of representations where none of the squares is divisible by 3, P_2 being the number of the remaining representations (all the squares but one being divisible by 3). We have

$$P_1 + P_2 = N(4n + x^2 + y^2 + z^2 + t^2),$$
$$\tfrac{3}{4}P_2 = N(4n = 9x^2 + y^2 + z^2 + t^2),$$

and therefore the required number of solutions will be found as soon as we can find P_1. But we can obtain all the solutions of the equation

$$4n = x^2 + y^2 + z^2 + t^2,$$

where none of the numbers x, y, z, t is divisible by 3, by means of the formulas

$$x = \xi + \eta, \quad y = \xi - \eta, \quad z = \zeta + \theta, \quad t = \zeta - \theta,$$

where ξ, η, ζ, θ range over all the solutions of the equation

$$2n = \xi^2 + \eta^2 + \zeta^2 + \theta^2,$$

subject to the condition that neither $\xi + \eta$ nor $\zeta + \theta$ is divisible by 3. But it is easy to see that there are exactly

$$\tfrac{2}{3}N(2n = \xi^2 + \eta^2 + \zeta^2 + \theta^2)$$

solutions of this kind. Therefore

$$P_1 = \tfrac{2}{3}N(n = x^2 + y^2 + z^2 + t^2)$$

and

$$P_2 = \tfrac{1}{3}N(n = x^2 + y^2 + z^2 + t^2);$$

whence

$$N(4n = 9x^2 + y^2 + z^2 + t^2) = \tfrac{1}{4}N(n = x^2 + y^2 + z^2 + t^2).$$

It is easy now to find in case of an *even* $n \equiv 1 \pmod 3$ the following expressions for our sums:

$$\sum F(n - 9h^2) = \tfrac{1}{2}\mathrm{X}(n/2), \tag{14}$$

$$\sum G(n - 9h^2) = \tfrac{1}{2}\Phi(n/2). \tag{15}$$

We encounter greater difficulties in finding the number of solutions of the equation

$$n = 9x^2 + y^2 + z^2 + t^2,$$

when n is *odd*. It follows from the preceding that the number of *all* solutions of the equation

$$4n = 9x^2 + y^2 + z^2 + t^2$$

is $6\Phi(n)$ so that it remains to find the number of solutions of the equation

$$4n = 9i^2 + j^2 + k^2 + l^2$$

in *odd* numbers i, j, k, l. For this purpose we write first the equation (3) in the form

$$\sum [3G(n-9h^2) - 2F(n-9h^2)] = \tfrac{1}{4}N(n = 9x^2 + y^2 + 3z^2 + 3t^2)$$

and, this being true for every $n \equiv 1 \pmod 3$, it follows

$$N(n = y^2 + 3z^2 + 3t^2) = 4[3G(n) - 2F(n)], \quad n \equiv 1 \pmod 3.$$

If, in addition to this $n \equiv 3 \pmod 8$, then $G(n) = \tfrac{4}{3}F(n)$ and

$$N(n = y^2 + 3z^2 + 3t^2) = 8F(n)$$

or, what is the same,

$$N(n = y^2 + z^2 + t^2) = N(n = y^2 + 3z^2 + 3t^2).$$

By means of this important relation we find

$$\begin{aligned} R &= N(4n = 9i^2 + j^2 + k^2 + l^2) \\ &= N(4n = 9i^2 + j^2 + 3k^2 + 3l^2;\ i \text{ odd},\ j \text{ even}). \end{aligned}$$

Now, the total number of solutions of the equation

$$4n = 9x^2 + y^2 + 3z^2 + 3t^2,$$

where x and y are of different parity, being $8\Phi(n)$ it remains to find the sum

$$S = \sum (-1)^x$$

extended over all the solutions of the preceding equation with *odd* value of $x + y$ and this can be done by means of the equation (28) of our paper "On Jacobi's arithmetical theorems, etc." which in the present case may be exhibited as follows:

$$\sum (-1)^x = \{4s(s/3)\}, \quad k = s^2, \quad s > 0$$

the sum being extended over the solutions of the equation

$$4k = 9x^2 + y^2 + 3z^2 \text{ with } z \textit{ odd}.$$

Applying this result we find

$$S = 8\sum (s/3)s = 8\sigma$$

where the summation is extended over all solutions of the indeterminate equation

$$n = s^2 + 3t^2, \quad s > 0.$$

The sum S being thus determined, we get

$$R = 4\Phi(n) - 4\sigma,$$
$$N(n = 9x^2 + y^2 + z^2 + t^2) = 2\Phi(n) + 4\sigma,$$

and finally

$$\sum F(n - 9h^2) = \tfrac{1}{4}\Phi(n) + \tfrac{1}{4}\sigma, \tag{16}$$

$$\sum G(n - 9h^2) = \tfrac{1}{3}\Phi(n) + \tfrac{1}{6}\sigma. \tag{17}$$

It is hardly necessary to add that the relations we have established contain Gierster's relations of the third degree, which involve the numerical function

$$H(n) = G(n) - F(n)$$

representing the number of classes corresponding to a given negative *discriminant* $-n = b^2 - 4ac$.

7. *The equation derived from* (IV) *when* $f(x) = \cos(2\pi x/5)$. Before giving the fundamental equation we can derive from (IV) by taking $f(x) = \cos(2\pi x/5)$ it is necessary to recall certain known results concerning the number of representations by some quaternary forms. Let us denote by $N_1(n)$ and $N_1^*(n)$ respectively the numbers of solutions of the equations

$$n = x^2 + y^2 + z^2 + 5t^2, \quad n = x^2 + 5y^2 + 5z^2 + 5t^2.$$

Following expressions may be considered as known

$$N_1(n) = (-1)^n \sum (-1)^d [5(d'/5) + (d/5)]d,$$
$$N_1^*(n) = (-1)^n \sum (-1)^d [(d'/5) + (d/5)]d,$$

both sums being extended over conjugate divisions of n. Considering the equation

$$2n = x^2 + 2y^2 + 5z^2 + 10t^2$$

and denoting by $N_2(n)$ the number of its solutions we have

$$N_2(n) = (-1)^n 2 \sum (-1)^d (d/5)^2 d$$

Using now the expressions for $N_1(n)$, $N_1^*(n)$ and $N_2(n)$ it is easy to find the equation

$$\sum (-1)^d d[1 - \cos(2\pi d/5)] = (-1)^n [\tfrac{5}{8}N_2(n) + (5^{1/2}/16)N_1(n) - (5^{3/2}/16)N_1^*(n)]$$

holding true for every n. This equation will serve us to transform the right-hand number of (IV) where we assume $f(x) = \cos(2\pi x/5)$. Proceeding exactly as in § 3 we can obtain the following fundamental equation:

$$\sum [3G(n - h^2) - 2F(n - h^2)] \cos(4\pi h/5) = \tfrac{5}{4}\sigma_1 + \tfrac{5}{8}\sigma_2 - (5^{3/2}/8)\sigma_3 + 2\sigma_4$$

where $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ denote the following sums, the extent of summation being shown behind the expression of the corresponding sum:

$$\sigma_1 = \sum \cos(4\pi y/5), \quad n = x^2 + y^2 + z^2 + 5t^2$$
$$\sigma_2 = \sum \cos(4\pi x/5), \quad n = x^2 + y_2 + z^2 + 5t^2$$
$$\sigma_3 = \sum \cos(4\pi x/5), \quad n = x^2 + 5y^2 + 5z^2 + 5t^2$$
$$\sigma_4 = (-1)^n \sum (-1)^{d'}(d - d') \cos[(2\pi/5)(d' + d)], \quad n = dd', \quad d < d'.$$

Now, putting

$$\sigma_5 = \sum \cos(4\pi x/5), \quad n = x^2 + y^2 + z^2 + t^2$$

we have by Gauss theorem

$$\sum [2F(n - h^2) - G(n - h^2)] \cos(4\pi h/5) = 1/12\sigma_5$$

whence and from the preceding equation it follows

$$\text{(V)} \quad 4\sum F(n - h^2) \cos(4\pi h/5) = 5/4\sigma_1 + (5^{1/2}/8)\sigma_2 - (5^{3/2}/8)\sigma_3 + 1/4\sigma_5 + 2\sigma_4.$$

What remains now is to express different sums in the right-hand member of this equation by means of certain standard numerical functions. This can be easily done in case when $n \equiv 0 \pmod 5$ both the remaining cases $n \equiv \pm 1$, $n \equiv \pm 2 \pmod 5$ requiring more elaborate investigation.

8. *Discussion of the case* $n \equiv 0 \pmod 5$. First it is obvious that for n divisible by 5

$$\sigma_1 = N(2n = x^2 + 2y^2 + 5z^2 + 10t^2) = N_2(n).$$

Second, in this case it is easy to find the number of solutions of the equation

$$n = x^2 + y^2 + z^2 + 5t^2$$

where x satisfies one of the congruences: $x \equiv 0, \pm 1, \pm 2 \pmod 5$, and to compute the sums σ_2 and σ_3. We find

$$\sigma_2 - 5\sigma_3 = -4N_1^*(n).$$

As to the sum σ_4, it has different expressions according to the parity of n, namely for an *odd* n

$$\sigma_4 = 2^5/2\Psi(n/25) - 1/2\Psi(n) + (5^{1/2}/2)N_1^*(n) - 5^{1/2}\sum (d' + d)/5; \quad n = dd', \quad d < d',$$

while for an *even* n

$$\sigma_4 = 25\Psi(n/100) + {}^{25}\!/_2\Phi(n/50) - \Psi(n/4) - {}^1\!/_2\Phi(n/2) + (5^1\!/_2/2)N_1^*(n) + 2(5^1\!/_2)\sum[(d'+d)/5]d; \quad n = 4dd',\ d < d'.$$

To evaluate the sum σ_5 we notice first that in the equation

$$n = x^2 + y^2 + z^2 + t^2$$

either all numbers x, y, z, t are divisible by 5, or only two of them or none. Let us denote by Q the number of solutions where

$$x^2 \equiv 1, \quad y^2 \equiv -1, \quad z \equiv 0, \quad t \equiv 0 \pmod 5,$$

and by Q_1 the number of solutions where

$$x^2 \equiv 1, \quad y^2 \equiv 1, \quad z^2 \equiv -1, \quad t^2 \equiv -1 \pmod 5.$$

The obvious relation

$$12Q + 6Q_1 = N(n = x^2 + y^2 + z^2 + t^2) - N(n/25 = x^2 + y^2 + z^2 + t^2)$$

enables us to express Q_1 in terms of Q and known numerical functions. After a simple discussion we find

$$\sigma_5 = {}^{15}\!/_8 Q + {}^5\!/_2\Phi(n/25) - {}^1\!/_2\Phi(n) \text{ for an } \textit{odd } n,$$
$$\sigma_5 = {}^{15}\!/_8 Q + {}^{15}\!/_2\phi(n/25) - {}^3\!/_2\Phi(n) \text{ for an } \textit{even } n,$$

denoting by $\Phi(n)$ the sum of *odd* divisions of n.

Introducing these various expressions into the equation (V) and separating rational and irrational parts, we get first

$$(18) \qquad \sum (h/5)\, F(n - h^2) = \sum [(d'+d)/5]d, \quad n = dd', \quad d < d',$$

for an *odd* n and

$$(19) \qquad \sum (h/5)\, F(n - h^2) = -2\sum [(d'+d)/5]d, \quad n = 4dd', \quad d < d'$$

for an *even* n.

The equation obtained by equating rational parts, combined with Kronecker's relations

$$\sum F(n - h^2) = {}^1\!/_2\Phi(n) + {}^1\!/_2\Psi(n) \text{ for an odd } n,$$
$$\sum F(n - h^2) = \mathrm{X}(n) - \Phi(n/4) + \Psi(n/4) \text{ for an } \textit{even } n,$$

gives in case of an *odd* n

$$\sum F(n - 25h^2) = {}^1\!/_2[\Phi(n) - 5\Phi(n/5)] + {}^5\!/_2\Psi(n/25) + {}^1\!/_2\Phi(n/25) + {}^3\!/_8 Q.$$

But the sum

$$\tfrac{1}{2}\Phi(n/25) + \tfrac{3}{8}Q$$

represents 1/16 of all solutions of the equation

$$n = x^2 + y^2 + z^2 + 25t^2,$$

so that, denoting this number by $R(n)$, we have

$$\sum F(n - 25h^2) = \tfrac{1}{2}[\Phi(n) - 5\Phi(n/5)] + \tfrac{1}{16}R(n) + \tfrac{5}{2}\Psi(n/25)$$

or, which is the same,

$$(20) \quad \sum F(n - 25h^2) = \tfrac{1}{4}N(n) + \tfrac{1}{16}R(n) + \tfrac{5}{2}\Psi(n/25).$$

For an *even* n the expression for the corresponding sum is

$$(21) \quad \sum F(n - 25h^2) = \tfrac{1}{4}N_2(n) + \tfrac{1}{16}R(n) + \tfrac{5}{2}\Phi(n/50) + 5\Psi(n/100).$$

The equations (20) and (21) involve the numerical function $R(n)$ together with other simple functions depending only on divisors. It seems very difficult to reduce $R(n)$ *in case of an odd* n to some simpler functions; nevertheless the equation (20) even in this imperfect form may be of great use. For an *even* n it is possible to find a simple expression for $R(n)$. For an *even* n we have

$$\sum \cos(2\pi\xi/5) = \sum \cos(2\pi x/5) \cos(2\pi y/5)$$

the sums being extended over solutions of the equations

$$2n = \xi^2 + \eta^2 + \zeta^2 + \theta^2, \quad n = x^2 + y^2 + z^2 + t^2.$$

Now it is possible to express the left-hand member in terms of $Q(2n)$, $Q_1(2n)$ and the known numerical functions, and the right-hand member may be in the same way expressed by means of $Q(n)$ and $Q_1(n)$. Thus we get the relation

$$3Q(2n) - Q_1(2n) = -\tfrac{1}{3}Q(n) - \tfrac{1}{6}Q_1(n),$$

or, changing n into $2n$,

$$9Q(n) - 3Q_1(n) = -Q(2n) - \tfrac{1}{2}Q_1(2n).$$

But

$$-Q(2n) - \tfrac{1}{2}Q_1(2n) = -2[\phi(n) - \phi(n/25)];$$

therefore

$$3Q(n) + Q_1(n) = -\tfrac{2}{3}[\phi(n) - \phi(n/25)],$$

and

$$2Q(n) + Q_1(n) = 4[\phi(n) - \phi(n/25)];$$

whence it follows that

$$Q(n) = \tfrac{1}{4}Q_1(n) = \tfrac{2}{3}[\phi(n) - \phi(n/25)],$$

and

$$R(n) = 4[\phi(n) + 5\phi(n/25)] = 24\phi(n/5).$$

It remains to introduce this expression into (21) to get after all reductions the final expression for an *even* n:

$$(22) \quad \sum F(n - 25h^2) = \Phi(n/4) - 4\Phi(n/20) + 2\phi(n/5) + 5\Psi(n/100).$$

As soon as we have (20) and (22) it is easy to find the expressions for the sum

$$\sum F(n - h^2)$$

where h ranges separately either over quadratic residus or non-residus modulo 5. But it is hardly necessary to give final expressions. In the following reference will be made to the number of representation of an even number by the form

$$x^2 + y^2 + 5z^2 + 5t^2$$

which can be easily derived from what has been established. For, starting from the relation

$$N(5n = x^2 + y^2) + N(n/5 = x^2 + y^2) = 2N(n = x^2 + y^2)$$

we find first

$$2N(n = x^2 + y^2 + 5z^2)$$
$$= N(5n = x^2 + y^2 + 25z^2) + N(n/5 = x^2 + y^2 + z^2)$$

and then

$$2N(n = x^2 + y^2 + 5z^2 + 5t^2)$$
$$N(5n + x^2 + y^2 + 25z^2 + 25t^2) + N(n/5 = x^2 + y^2 + z^2 + t^2)$$

whatever n may be. Now supposing n *even* we have

$$N(5n = x^2 + y^2 + 25z^2 + 25t^2) = 2Q(5n) + 24\phi(n/5);$$

and using the preceding expression for $Q(5n)$,

$$N(n = x^2 + y^2 + 5z^2 + 5t^2) = \tfrac{2}{3}[\phi(5n) - \phi(n/5)] + 24\phi(n/5),$$

or supposing $n = 5^\beta m$, m non-divisible by 5,

$$N(n = x^2 + y^2 + 5z^2 + 5t^2) = 2[5^{\beta+1} - 3]\phi(m).$$

9. *Subsidiary developments.* Trying to evaluate the right-hand mem-

ber of (V) when $m \equiv \pm 1$ or $\equiv \pm 2 \pmod 5$, we meet with greater difficulties. These difficulties arise from the fact that the indeterminate equations

$$2n = x^2 + 2y^2 + 5z^2 + 10t^2,$$
$$n = x^2 + y^2 + z^2 + 5t^2,$$
$$n = x^2 + y^2 + z^2 + t^2,$$

possess different types of solutions and it does not seem easy to determine the number of solutions belonging to these various types. The equation

$$2n = x^2 + 2y^2 + 5z^2 + 10t^2,$$

supposing $n \equiv \pm 1$ or $\pm 2 \pmod 5$, may have solutions where x or y are divisible by 5. Let us denote the number of such solutions by $T(n)$, the number of the remaining solutions being denoted by $T'(n)$, so that

$$T(n) + T'(n) = N_2(n).$$

Similarly the equation

$$n = x^2 + y^2 + z^2 + 5t^2$$

may have solutions where none of the numbers x, y, z is divisible by 5, while in the remaining solutions one [in case $n \equiv \pm 2 \pmod 5$] or two of these numbers [in case $n \equiv \pm 1 \pmod 5$] are divisible by 5. Let us introduce the notation $U(n)$ for the number of solutions of the equation

$$n = x^2 + 25y^2 + 25z^2 + 5t^2$$

in case $n \equiv \pm 1 \pmod 5$, and of the equation

$$n = x^2 + y^2 + 25z^2 + 5t^2$$

in case $n \equiv \pm 2 \pmod 5$. Finally the equation

$$n = x^2 + y^2 + z^2 + t^2$$

possesses three different types of solutions, namely, solutions where none of the numbers x, y, z, t is divisible by 5, or only one of them or more than one. Corresponding to the case $n \equiv \pm 1 \pmod 5$ $Q(n)$, $Q_1(n)$, $Q_2(n)$ will denote the number of solutions satisfying the respective conditions:

$$\left.\begin{array}{llll} x^2 \equiv \mp 1, & y^2 \equiv \mp 1, & z^2 \equiv \mp 1, & t^2 \equiv \mp 1 \\ x^2 \equiv \pm 1, & y^2 \equiv \pm 1, & z^2 \equiv \mp 1, & t^2 \equiv 0 \\ x^2 \equiv \pm 1, & y^2 \equiv 0, & z^2 \equiv 0, & t^2 \equiv 0 \end{array}\right\} \pmod 5$$

the upper and lower signs corresponding to those in the congruence

$n \equiv \pm 1 \pmod 5$. In the case $n \equiv \pm 2 \pmod 5$ the same notations will be used to denote the number of solutions satisfying the conditions

$$\left.\begin{array}{llll} x^2 \equiv \pm 1, & y^2 \equiv \pm 1, & z^2 \equiv \pm 1, & t^2 \equiv \mp 1 \\ x^2 \equiv \mp 1, & y^2 \equiv \mp 1, & z^2 \equiv \mp 1, & t^2 \equiv 0 \\ x^2 \equiv \pm 1, & y^2 \equiv \pm 1, & z^2 \equiv 0, & t^2 \equiv 0 \end{array}\right\} \pmod 5.$$

For an *even* n we have the relation

$$Q(n) + 12Q_1(n) + 4Q_2(n) = 24\phi(n) \tag{23}$$

when $n \equiv \pm 1 \pmod 5$ or the relation

$$2Q(n) + 2Q_1(n) + 3Q_2(n) = 12\phi(n) \tag{24}$$

when $n \equiv \pm 2 \pmod 5$. Moreover, starting from the relation $N(n = x^2 + 5y^2 + 5z^2) = \frac{1}{3}N(n = x^2 + y^2 + z^2)$ when $n \equiv \pm 2 \pmod 5$ which can be easily verified, we find

$$Q_1(n) + Q_2(n) = \tfrac{1}{2}N(n = x^2 + y^2 + 5z^2 + 5t^2) \text{ when } n \equiv \pm 1 \pmod 5$$
$$Q(n) + Q_2(n) = N(n = x^2 + y^2 + 5z^2 + 5t^2) \text{ when } n \equiv \pm 1 \pmod 5$$

so that for an *even* n

$$Q_1(n) + Q_2(n) = 2\phi(n), \quad n \equiv \pm 1 \pmod 5$$
$$Q(n) + Q_2(n) = 4\phi(n), \quad n \equiv \pm 2 \pmod 5.$$

Combined with the preceding relations these relations permit to express $Q(n)$ and $Q_1(n)$ in terms of $Q_2(n)$ when n is even, but $Q_2(n)$ remains unknown, and it seems very difficult to reduce it to some simpler numerical functions. For an *even* n we have the obvious equation

$$\sum \cos(2\pi x/5) = \sum \cos(2\pi\xi/5) \cos(2\pi\eta/5)$$

the sums being extended over solutions of the respective equations

$$2n = x^2 + y^2 + z^2 + t^2, \quad n = \xi^2 + \eta^2 + \zeta^2 + \theta^2.$$

Now it is possible to express the left-hand member of the preceding equation as a linear function of $Q(2n)$, $Q_1(2n)$, $Q_2(2n)$ while the right-hand member is a linear function of $Q(n)$, $Q_1(n)$, $Q_2(n)$. Equating the rational and irrational parts, we get two linear equations from which we can eliminate $Q(2n)$ by means of (23) or (24) and solve the resulting equations for $Q_1(2n)$, $Q_2(2n)$.

This way leads to the following results: for an *even* $n \equiv \pm 1 \pmod 5$

$$Q(2n) = 2Q_1(n) = 4\phi(n) - 2Q_2(n),$$
$$Q_1(2n) = Q_1(n) = 2\phi(n) - Q_2(n),$$
$$Q_2(2n) = 4\phi(n) - 2Q_1(n) = 2Q_2(n),$$

while for an *even* $n \equiv \pm 2 \pmod 5$

$$Q(2n) = 4Q_2(n),$$
$$Q_1(2n) = 2\phi(n) - \tfrac{1}{2}Q_2(n),$$
$$Q_2(2n) = \tfrac{1}{2}Q_2(n).$$

The equation connecting $Q_2(2n)$ and $Q_2(n)$ may be also written as follows:

(25) $$Q_2(2n) = \tfrac{1}{4}[5 + 3(n/5)]Q_2(n)$$

provided n is even non-divisible by 5. In a similar way we can find the expressions for $Q_1(4n)$, $Q_2(4n)$ in terms of $Q_1(n)$, $Q_2(n)$ for an *odd* n. The final expressions are as follows: for $n \equiv \pm 1 \pmod 5$

$$Q_1(4n) = 3Q_1(n), \quad Q_2(4n) = 2\phi(n) - 3Q_1(n),$$

and for $n \equiv \pm 2 \pmod 5$

$$Q_1(4n) = 2\phi(n) - \tfrac{3}{2}Q_2(n), \quad Q_2(4n) = 3Q_2(n).$$

10. *Expression for $T(n)$.* Our next aim is to obtain an expression for $T(n)$ involving only elementary functions and $Q_2(n)$. For this purpose we make use of a certain fundamental identity which may be found in our paper "Sur les relations entre les nombres des classes des formes quadratiques binaires et positives. Premier mémoire" as well as in the paper "On Jacobi's arithmetical theorems, etc." Let $F(x, y, z)$ denote an arbitrary function defined for integral values of x, y, z and satisfying the conditions:

$$F(-x, y, z) = -F(x, y, z); \quad F(x, -y, -z) = F(x, y, z);$$
$$F(0, y, z) = 0.$$

For such a function we have the following important relation

(VI) $$\sum F(\Delta + \Delta', h, \Delta - \Delta') = 2 \sum F(\delta - 2i, d + i, 2d + 2i + \delta) + T$$

where the sums are extended over solutions of the indeterminate equations

$$n = h^2 + \Delta\Delta', \quad n = i^2 + d\delta;$$

here h and i may have any integral values, while Δ, Δ', d, δ are supposed to

be positive integers. As to the term T it is zero except when n is a perfect square $= s^2$ in which case

$$T = \sum_{j=1}^{2s-1} [F(2s, s-j, 2s-2j) - 2F(2s-j, s, 2s-j)].$$

By means of a simple device it is possible to obtain another relation which may be useful on many occasions. Denoting by p any odd number which is prime or composed of different primes, let us take pn instead of n in (VI) and furthermore let us suppose that $F(x, y, z) = 0$ whenever

$$2y - z \not\equiv 0 \text{ or } y \equiv 0 \pmod{p}$$

which is consistent with the conditions imposed on $F(x, y, z)$. Under these circumstances the right-hand member of (VI) becomes

$$2 \sum F(p\delta - 2pi, d + pi, 2d + 2pi - p\delta) + T'$$

where the sum is now extended over solutions of the equation

$$n = pi^2 + d\delta,$$

d being not divisible by p and $T' = 0$ unless $n = ps^2$ in which case

$$T' = \sum F(ps, ps - j, 2ps - 2j),$$

j ranging over numbers of the series 1, 2, 3, $\cdots$ which are *not divisible* by p. The left-hand member sum is now extended over solutions of the equation

$$pn = h^2 + \Delta\Delta',$$

subject to the conditions

$$h \not\equiv 0, \quad 2h + \Delta' - \Delta \equiv 0 \pmod{p},$$

which are entirely equivalent to

$$h \not\equiv 0, \quad h \equiv \Delta \pmod{p}.$$

Now we can disregard the condition $h \equiv \Delta \pmod{p}$ if we introduce the function $\omega(h, \Delta)$ whose value is 0 unless $h \equiv \Delta \pmod{p}$, in which case $\omega(h, \Delta) = 1$. Such a function is given, for instance, by

$$\omega(h, \Delta) = 1/p \left[1 + 2\cos[2\pi(h-\Delta)/p] + 2\cos[4\pi(h-\Delta)/p] + \cdots + 2\cos\{[(p-1)/2]\, 2\pi(h-\Delta)/p\}\right].$$

We can represent now the right-hand member as follows

$$\sum \omega(h, \Delta)\, F(\Delta + \Delta', h, \Delta - \Delta')$$

where the sum is extended over solutions of the equation

$$pn = h^2 + \Delta\Delta'$$

with *h non-divisible by p* and the function $F(x, y, z)$ being of the same general character as in (VI). This way we finally arrive at the important relation

$$\text{(VII)} \quad \sum \omega(h, \Delta)\, F(\Delta + \Delta', h, \Delta - \Delta') = 2 \sum F(p\delta - 2pi,\ d + pi,\ 2d + 2pi - p\delta + T'$$

the extent of summations being explained above. If we suppose that instead of the single condition

$$F(x, -y, z) = F(x, y, z)$$

we have two conditions

$$F(x, -y, z) = F(x, y, -z) = F(x, y, z)\ ,$$

we can take simply

$$\omega(h, \Delta) = 1/p \left[1 + 2\cos(2\pi h/p)\cos(2\pi\Delta/p) + 2\cos(4\pi h/p)\cos 4\pi\Delta/p) + \cdots + 2\cos\{[(p-1)/2]\, 2\pi h/p\}\cos\{[(p-1)/2]\, 2\pi\Delta/p\}\right].$$

Taking $p = 3$ we have

$$\omega(h, \Delta) = \tfrac{1}{2}$$

so that for any function $F(x, y, z)$ odd with respect to x and even with respect to y and z we have

$$\text{(VIII)} \quad \tfrac{1}{2} \sum F(\Delta + \Delta',\ h,\ \Delta - \Delta') = 2 \sum F(3\delta - 6i,\ d + 3i,\ 2d + 6i - 3\delta) + T'$$

sums being extended over solutions of the equations with h and d not divisible by 3. For $p = 5$ we have

$$\omega(h, \Delta) = \tfrac{1}{4}\,[1 + (h\Delta/5)]$$

and the following relation holds true for every function $F(x, y, z)$ odd with respect to x and even with respect to y and z:

$$\text{(IX)} \quad \tfrac{1}{4} \sum [1 + (h\Delta/5)]\, F(\Delta + \Delta',\ h,\ \Delta - \Delta') = 2 \sum F(5\delta - 10i,\ d + 5i,\ 2d + 10\text{i} - 5\delta) + T'$$

sums being extended over solutions of the equations

$$5n = h^2 + \Delta\Delta', \quad n = 5i^2 + d\delta$$

with h and d not divisible by 5. The equation (IX) is of fundamental importance for what follows. We assume in it

$$F(x, y, z) = (y/5)\sin(\pi x/4)\cos(\pi z/4)$$

and suppose $n \equiv \pm 2 \pmod 5$. The result of this substitution is

$$(26)\quad \tfrac{1}{4}\sum (h/5)\sin(\pi\Delta/2) + \tfrac{1}{4}\sum(\Delta/5)\sin(\pi\Delta/2)$$
$$= \sum (d/5)\sin(\pi d/2) - 2\sum(-1)^{\delta}\cos(\pi d/2)(\delta/5)\sin(\pi\delta/2).$$

Now the sum

$$\sum \sin(\pi d/2)$$

extended over divisors of any number m is equal to one-fourth of the number of representations of m by the form $x^2 + y^2$. Similarly, the sum

$$\sum (d/5)\sin(\pi d/2)$$

extended over divisors of m is equal to one-half of the number of representations of m by two forms: $x^2 + 5y^2$ and $2x^2 - 2xy + 3y^2$ the first being unable to represent numbers $\equiv \pm 2$ and the second those $\equiv \pm 1 \pmod 5$. The first sum in the left-hand member of (26) may be, therefore, represented as the sum

$$\tfrac{1}{16}\sum (h/5)$$

extended over solutions of the equation

$$5n = h^2 + k^2 + l^2$$

with h not divisible by 5. But it is obvious that such a sum has zero for its value. The second sum in the left-hand member represents $\frac{1}{8}$ of the number of solutions of the equation

$$5n = h^2 + k^2 + 5l^2$$

with h not divisible by 5 and this is equal to

$$\tfrac{1}{4}N(n = x^2 + y^2 + z^2).$$

The first sum in the right-hand member of (26) is

$$\tfrac{1}{2}N(n = 5i^2 + 2j^2 + 2jk + 3k^2),$$

while the second sum has different expressions for an *odd* or *even* n, namely

$$-(-1)^{n-1/2}N(n = 5i^2 + 2j^2 - 2jk + 3k^2,\ i \text{ odd}),\ n \textit{ odd}$$
$$(-1)^{n/2}N(n = 5i^2 + 2j^2 - 2jk + 3k^2,\ i \text{ even}),\ n \textit{ even}$$

or, which is the same,

$$-(-1)^{n-1/2}N(n = 5i^2 + 2j^2 + 10k^2) \text{ for an } \textit{odd } n,$$
$$(-1)^{n/2}N(n = 5i^2 + 2j^2 + 10k^2) \text{ for an } \textit{even } n.$$

After substituting these various expressions into (26) we finally get

$$(27)\quad N(2n = x^2 + 5y^2 + 10z^2) + 2(-1)^{n-1/2} N(n = 2x^2 + 5y^2 + 10z^2) = \tfrac{1}{2}N(n = x^2 + y^2 + z^2)$$

$$(28)\quad N(2n = x^2 + 5y^2 + 10z^2) - 2(-1)^{n/2} N(n = 2x^2 + 5y^2 + 10z^2) = \tfrac{1}{2}N(n = x^2 + y^2 + z^2)$$

for an *odd* and *even* $n \equiv \pm 2 \pmod 5$ respectively. Let now n denote any number not divisible by 5. We substitute $2n - t^2$ instead of n in (27) and (28), choosing $t \equiv 0 \pmod 5$ when $n \equiv \pm 1$, $t \equiv \pm 1$ when $n \equiv 2$ and $t \equiv \pm 2$ when $n \equiv -2 \pmod 5$. As a result we obtain a single equation independently of whether t is odd or even:

$$N(4n - 2t^2 = x^2 + 5y^2 + 10z^2) - 2(-1)^n N(2n - t^2 = 2x^2 + 5y^2 + 10z^2) = \tfrac{1}{2} N(2n - t^2 = x^2 + y + z^2).$$

We let t now assume all the values, subject to the preceding limitations, which make $2n - t^2$ positive, and add all the resulting equations. What we get this way, attention paid to the definition of $T(n)$, $T'(n)$, $Q(n)$, $Q_1(n)$, $Q_2(n)$ given in § 9, may be presented as follows:

$$T(2n) - (-1)^n 2T(n) = \tfrac{1}{2}(3Q_2(2n) + Q_1(2n)) \text{ in case } n \equiv \pm 1 \pmod 5,$$

$$T'(2n) - (-1)^n 2T'(n) = \tfrac{1}{2}(Q(2n) + 3Q_1(2n)) \text{ in case } n \equiv \pm 2 \pmod 5.$$

Now we can express $T'(2n)$, $T'(n)$ by $T(2n)$, $T(n)$; and $Q(2n)$, $Q_1(2n)$ by $Q_2(2n)$. All simplifications performed, we obtain:

$$\begin{aligned}
T(2n) - 2T(n) &= \phi(n) + \tfrac{5}{2}Q_2(n), && n \equiv \pm 1 \pmod 5, && n \text{ even}\\
T(2n) + 2T(n) &= \phi(n) + \tfrac{5}{4}Q_2(2n), && n \equiv \pm 1 \pmod 5, && n \text{ odd},\\
T(2n) - 2T(n) &= 3\phi(n) - \tfrac{5}{4}Q_2(n), && n \equiv \pm 2 \pmod 5, && n \text{ even},\\
T(2n) + 2T(n) &= 3\phi(n) - \tfrac{5}{2}Q_2(2n), && n \equiv \pm 2 \pmod 5, && n \text{ odd},
\end{aligned}$$

which also may be written as follows:

$$(29)\quad T(2n) - 2T(n) = [2 - (n/5)]\phi(n) + \tfrac{5}{8}[1 + 3(n/5)]Q_2(n), \qquad n \text{ even};$$

$$(30)\quad T(2n) + 2T(n) = [2 - (n/5)]\phi(n) - \tfrac{5}{8}[1 - 3(n/5)]Q_2(2n), \qquad n \text{ odd}.$$

To these equations we can add (25):

$$Q_2(2n) = \tfrac{1}{4}[5 + 3(n/5)]Q_2(n), \quad n \text{ even}.$$

Now it is easy to see that for an odd n

$$T(2n) = T(n),$$

and therefore we can find from (30) the expression for $T(2n)$, n being odd. Using then (29) and (25) we can find the expression for $T(n)$, n being any *even* number $= 2^k m$, m odd. The final results are as follows:

(31) $T(n) = [2^k - 2 + (2^k + 1/3)]\phi(n) - \frac{5}{6}Q_2(n)$, $n \equiv \pm 1 \pmod 5$,

(32) $T(n) = [2^k - 2 + 2^k - 1/3)]\phi(n) + \frac{5}{12}Q_2(n)$, $n \equiv \pm 2 \pmod 5$.

As to *odd* values of n, we have

(33) $T(n) = 2\phi(n) - \frac{5}{2}Q_1(n)$, $n \equiv \pm 1 \pmod 5$,
$n \equiv \pm 1 \pmod 5$,

(34) $T(n) = \phi(n) - \frac{5}{4}Q_2(n)$, $n \equiv \pm 2 \pmod 5$.

11. *Expression for $U(n)$.* Let $f(x)$ denote an odd function and n a number non-divisible by 5. Taking in (VI)

$F(x, y, z) = 0$ when $x \equiv 0 \pmod 2$ or $y \not\equiv 0 \pmod 5$,
$F(x, y, z) = f(x)$ otherwise,

we obtain

$$\sum f(\Delta + \Delta') = \sum f(\delta - 2i)$$

where the first sum is extended over solutions of the equation

$$n = 25h^2 + \Delta\Delta'$$

with *odd* Δ and *even* Δ' while the second over solutions of

$$n = i^2 + d\delta$$

with *odd* δ and $i + d$ divisible by 5. In the same way as before, denoting by $\omega(d, i)$ the sum

$$\omega(d, i) = \tfrac{1}{5}\left[1 + 2\cos\left[2\pi(d + i)/5 + 2\cos[4\pi(d + i)/5]\right]\right],$$

we can write the preceding equation as follows

$$\sum f(\Delta + \Delta') = \sum \omega(d, i)\, f(\delta - 2i)$$

the right-hand member sum this time being extended over solutions of

$$n = i^2 + d\delta$$

with δ *odd* and i and d non-divisible by 5. Now we take

$$f(x) = (-1)^{(x-1)/2}$$

and replace n by $4n$; after evident simplifications we get

$$(35) \qquad \sum (-1)^{(\Delta-1)/2} = \tfrac{1}{4} \sum [1 + (id/5)]\,(-1)^{(\delta-1)/2+i}$$

the sums being extended over solutions of the equations

$$n = 25h^2 + \Delta\Delta', \ \Delta \text{ odd},$$
$$4n = i^2 + d\delta, \ \delta \text{ odd}, \ i \text{ and } d \not\equiv 0 \pmod 5.$$

Now for any number m

$$\sum (-1)^{(\delta-1)/2} = \tfrac{1}{4} N(m = x^2 + y^2)$$
$$\sum (-1)^{(\delta-1)/2}(d/5) = \tfrac{1}{2}[N(m = x^2 + 5y^2) - N(2m = x^2 + 5y^2)]$$

both sums being extended over all the representations of m in the form

$$m = d\delta, \quad \delta \text{ odd}$$

whence it is easy to see that the relation (35) implies the following equations:

$$5N(n = x^2 + y^2 + 25z^2) - N(n = x^2 + y^2 + z^2)$$
$$= \mp 2N(n = x^2 + y^2 + 5z^2) - 2C, \quad n \equiv \pm 2 \pmod 5,$$
$$5N(n = x^2 + 25y^2 + 25z^2) - \tfrac{1}{3}N(n = x^2 + y^2 + z^2)$$
$$= \mp \tfrac{1}{2}N(n = x^2 + y^2 + 5z^2) - C, \quad n \equiv \pm 1 \pmod 5,$$

where C stands for the sum

$$\sum (-1)^i (i/5)$$

extended over all solutions of the equation

$$8n = 2i^2 + j^2 + 5k^2 \ .$$

Let us introduce now the sums

$$A = \sum_{(a)} (-1)^i, \qquad B = \sum_{(b)} (-1)^i$$

extended over solutions of the same equation subject to the following limitations:

(a) $i^2 \equiv \pm 1$ in case $n \equiv \pm 1$; $\quad i \equiv 0$ in case $n \equiv \pm 2 \pmod 5$,
(b) $i^2 \equiv \mp 1$ in case $n \equiv \pm 1$; $\quad i^2 \equiv \pm 1$ in case $n \equiv \pm 2 \pmod 5$.

It is obvious that

$$\pm C = A - B \text{ in case } n \equiv \pm 1 \pmod 5,$$
$$\pm C = \quad B \text{ in case } n \equiv \pm 2 \pmod 5.$$

There exists a relation between A and B which can be easily derived from (IX). To obtain this relation we suppose for a moment that n is any num-

ber, then take $4n$ instead of n in (IX) and define the function $F(x,y,z)$ as follows:

$$F(x,y,z) = 0 \text{ for even } x,$$
$$F(x,y,z) = \sin \pi x/4 \text{ for odd } x.$$

A simple discussion of the relations thus obtained leads to the following result:

$$[(-1)^n/4] \sum_{(a)} (-1)^h - [(-1)^n/4] \sum_{(b)} (-1)^h (h/5) = -\sum_{(c)} (-1)^z + \sum_{(d)} (-1)^z,$$

the extent of various summations being as follows:

(a) $5n = h^2 + k^2 + 2l^2$; $h \not\equiv 0 \pmod 5$
(b) $5n = h^2 + k^2 + 10l^2$; $h \not\equiv 0 \pmod 5$
(c) $n = x^2 + 2y^2 + 5z^2$
(d) $n/5 = x^2 + 2y^2 + z^2$.

Without discussing all possible cases we confine ourselves to considering only two of them. First suppose n *odd* and non-divisible by 5. In this case we have obviously

$$\sum_{(a)} (-1)^h = 0, \qquad \sum_{(b)} (-1)^z = 0$$

and the resulting equation

$$\sum_{(c)} (-1)^z = -\tfrac{1}{4} \sum_{(d)} (-1)^h (h/5)$$

being true for all odd n non-divisible by 5 leads to the remarkable conclusion:

$$\Sigma (-1)^z = -\tfrac{1}{4} \Sigma (-1)^h (h/5)$$

sums now being extended over solutions of the equations

$$n = x^2 + 5z^2, \quad 5n = h^2 + k^2$$

respectively. This theorem belongs to the same type as those well known theorems given by Jacobi.

Next we replace n by $8n$. This time we have

$$\sum_{(b)} (-1)^h (h/5) = 0$$

and the final result may be expressed thus:

$$(36) \quad \Sigma (-1)^z = \tfrac{5}{4} N(n/5 = x^2 + y^2 + z^2) - \tfrac{1}{4} N(5n = x^2 + y^2 + z^2)$$

or, which is the same,

$$\Sigma (-1)^v = -\tfrac{3}{2} N(n = x^2 + y^2 + 5z^2) + 2N(n/5 = x^2 + y^2 + z^2)$$

the sum being extended over all solutions of the equation

$$8n = x^2 + 2y^2 + 5z^2.$$

Supposing now again n non-divisible by 5 we see that the last equation is equivalent to

$$A + B = -\tfrac{3}{2}N(n = x^2 + y^2 + 5z^2).$$

Thus we can express B by A and finally arrive at the following relations interesting for many reasons:

$$(37)\quad \pm [5N(n = x^2 + y^2 + 25z^2) - N(n = x^2 + y^2 + z^2)] = N(n = x^2 + y^2 + 5z^2) + 2A, \quad n \equiv \pm 2 \pmod 5,$$

$$(38)\quad \pm [5N(n = x^2 + 25y^2 + 25z^2) - \tfrac{1}{3}N(n = x^2 + y^2 + z^2)] = -2N(n = x^2 + y^2 + 5z^2) - 2A, \quad n \equiv \pm 1 \pmod 5.$$

It follows from these relations:

$$(39)\quad \pm [5U(n) - N_1(n)] = n = x^2 + y^2 + 5z^2 + 5t^2) + 2W, \quad n \equiv \pm 2 \pmod 5$$

$$(40)\quad \pm [5U(n) - \tfrac{1}{3}N_1(n)] = -2N(n = x^2 + y^2 + 5z^2 + 5t^2) - 2W, \quad n \equiv \pm 1 \pmod 5$$

where W stands for the sum

$$W = \sum (-1)^t$$

extended over solutions of the equation

$$8n = 2i^2 + j^2 + 5k^2 + 40l^2,$$

subject to the following limitations:

$$i^2 \equiv \pm 1 \text{ when } n \equiv \pm 1 \pmod 5,$$
$$i^2 \equiv 0 \quad \text{when } n \equiv \pm 2 \pmod 5.$$

All which remains now is to find the appropriate expression for W. To this end we resort again to the equations (27), (28) in which we replace n by $4n - t^2$, multiply both parts by $(-1)^t$ and let t assume all integral values making $4n - t^2$ positive and satisfying the conditions

$$t \equiv 0 \quad \text{when } n \equiv \pm 2 \pmod 5,$$
$$t^2 \equiv \pm 1 \text{ when } n \equiv \pm 1 \pmod 5.$$

Adding all the equations thus obtained and performing obvious transformations, we finally find

$$W = 2T(n) - \tfrac{1}{2}T(4n) + \tfrac{1}{4}\sum(-1)^t, \quad n \equiv \pm 2 \pmod 5,$$
$$W = 2T'(n) - \tfrac{1}{2}T'(4n) + \tfrac{1}{4}\sum(-1)^t, \quad n \equiv \pm 1 \pmod 5,$$

the sum being extended over solutions of the equation

$$4n = x^2 + y^2 + z^2 + t^2,$$

where

$$t \equiv 0 \quad \text{when } n \equiv \pm 2 \pmod 5,$$
$$t^2 \equiv \pm 1 \text{ when } n \equiv \pm 1 \pmod 5.$$

The latter sum can be expressed in terms of $Q_1(n)$, $Q_2(n)$, the expression for $T(n)$ is given by (31), (32), (33), (34) and that for $T'(n)$ can easily be derived. So we can find the expression for W in terms of $Q_1(n)$, $Q_2(n)$ and elementary numerical functions; substituting it into (39) and (40) we find the following expressions for $U(n)$:

(41) $\pm 5U(n) = -2\phi(n) + \tfrac{5}{2}Q_2(n) \pm N_1(n), \quad n \equiv \pm 2 \pmod 5$, *n even*

(42) $\pm 5U(n) = 2\phi(n) - 1\tfrac{5}{2}Q_2(n) \pm N_1(n), \quad n \equiv \pm 2 \pmod 5$, *n odd*

(43) $\pm 5U(n) = 2\phi(n) - 5Q_2(n) \pm \tfrac{1}{3}N_1(n), \quad n \equiv \pm 1 \pmod 5$, *n even*

(44) $\pm 5U(n) = 8\phi(n) - 15Q_1(n) \pm \tfrac{1}{3}N_1(n), \quad n \equiv \pm 1 \pmod 5$, *n odd.*

12. *Discussion of the case* $n \equiv \pm 2 \pmod 5$. The really difficult part of all this investigation was in finding the expressions for $T(n)$ and $U(n)$. Now that this has been done the rest does not present difficulties any more. Returning to the equation (V) and considering first the case $n \pm \equiv 2 \pmod 5$ we obtain two relations by equating rational and irrational parts. The equation obtained by equating rational parts combined with Kronecker's expression for the sum

$$\sum F(n - h^2)$$

gives in all cases when $n \equiv \pm 2 \pmod 5$:

$$\sum F(n - 25h^2) = \tfrac{1}{2}\omega(n) + \tfrac{1}{16}(Q_1(n) + 3Q_2(n)) - \tfrac{1}{16}N_2(n) \pm \tfrac{1}{32}N_1(n)$$

where

$$\omega(n) = \tfrac{5}{16}(2T(n) \mp U(n)).$$

The equation resulting by equating irrational parts is

$$\pm \sum (h/5)\, F(n - h^2) = \tfrac{1}{2}\omega(n) \pm \tfrac{1}{32}N_1(n) - \tfrac{5}{16}N_2(n) + \tfrac{1}{16}(2Q(n) + 3Q_2(n) - 3Q_1(n)) + \tfrac{1}{2}\phi(n/2) + \Psi(n/4)$$

for an *even* n and

$$\pm \sum (h/5)\, F(n-h^2) = \tfrac{1}{2}\omega(n) \pm \tfrac{1}{32}N_1(n) - \tfrac{5}{16}N_2(n) + \tfrac{1}{16}(2Q(n) + 3Q_2(n) - 3Q_1(n)) + \Psi(n)$$

for an *odd* n. Now we can eliminate $Q(n)$ in any case, express $Q_1(n)$ by $Q_2(n)$ in case of an *even* n and substitute for $T(n)$, $U(n)$, $N_2(n)$ their expressions. This way we finally get

$$\sum F(n-25h^2) = \tfrac{1}{3}\Phi(n/4) + \tfrac{1}{6}\phi(n) + \tfrac{5}{24}Q_2(n)$$

when n is even and

$$\sum F(n-25h) = \tfrac{1}{8}\Phi(n) + [2Q_1(n) + Q_2(n)]/32$$

when n is odd. Similarly,

$$\pm \sum (h/5)\, F(n-h^2) = \Psi(n/4) - \tfrac{2}{3}\Phi(n/4) + \tfrac{1}{6}\phi(n) + \tfrac{5}{24}Q_2(n)$$

when n is even and

$$\pm \sum (h/5)\, F(n-h^2) = \tfrac{1}{2}\Psi(n) - \tfrac{1}{8}\Phi(n) - \tfrac{5}{32}(2Q_1(n) + Q_2(n))$$

when n is odd.

To obtain Gierster's relations involving only elementary functions depending on divisors, we notice that for an *even* n

$$\sum [2F(n-25h^2) - G(n-25h^2)] = \tfrac{1}{6}\phi(n) + \tfrac{5}{24}Q_2(n),$$
$$\pm \sum (h/5)\, [2F(n-h^2) - G(n-h^2)] = \tfrac{1}{6}\phi(n) + \tfrac{5}{24}Q_2(n),$$

so that, putting

$$H(n) = G(n) - F(n),$$

we have

$$\pm \sum (h/5)\, H(n-h^2) = \Psi(n/4) - \tfrac{2}{3}\Phi(n/4)$$
$$\sum H(n-25h^2) = \tfrac{1}{3}\Phi(n/4)$$

whence, changing n into $4n$ we finally get

$$\sum H(4n-25h^2) = \tfrac{1}{3}\Phi(n)$$

and

$$\sum_{h \equiv \pm 1 \pmod 5} H(4n-h^2) = \tfrac{2}{3}\Phi(n)$$

$$\sum_{h \equiv \pm 2 \pmod 5} H(4n-h^2) = \Psi(n)$$

for $n \equiv 2 \pmod 5$, while

$$\sum_{h \equiv \pm 1 \pmod 5} H(4n - h^2) = \Psi(n)$$

$$\sum_{h \equiv \pm 2 \pmod 5} H(4n - h^2) = \tfrac{2}{3}\Phi(n)$$

for $n \equiv -2 \pmod 5$.

13. *Discussion of the case* $n \equiv \pm 1 \pmod 5$. The equation obtained by equating rational parts in (V) combined with Kronecker's expression for the sum

$$\sum F(n - h^2)$$

yields in this case two relations, which, introducing two functions $\Psi_{+1}(n)$ and $\Psi_{-1}(n)$ defined by

$$\Psi_{+1}(n) = \sum [d' + (d/5)d], \quad n = dd', \ d < d'$$
$$\Psi_{-1}(n) = \sum [d' - (d/5)d], \quad n = dd', \ d < d',$$

may be presented as follows

$$\sum F(n - 25h^2) = \tfrac{1}{4}\mathrm{X}(n) + \Psi(n/4) - \Psi_{\pm 1} \quad (n/4)$$

for an *even* $n \equiv \pm 1 \pmod 5$, and

$$\sum F(n - 25h^2) = \tfrac{1}{8}\Phi(n) + \tfrac{1}{2}\Psi(n) - \tfrac{1}{2}\Psi_{\mp 1} \quad (n) + \tfrac{3}{16}(Q_1(n) + Q_2(n))$$

for an *odd* $n \equiv \pm 1 \pmod 5$. Similarly, by equating irrational parts we get

$$\pm \sum (h/5)\, F(n - h^2) = \omega(n) + \Psi(n/4) - \Psi_{\mp 1} \quad (n/4)$$
$$+ \tfrac{5}{16}(3Q_1(n) + Q_2(n) - \tfrac{3}{16}N_2(n) - \tfrac{3}{2}\phi(n) \pm \tfrac{1}{48}N_1(n)$$

for an *even* n, while

$$\pm \sum (h/5)\, \mathrm{F}(n - h^2) = \omega(n) - \Phi(n) + \tfrac{1}{2}\Psi_{\mp 1} \quad (n)$$
$$+ \tfrac{5}{16}(3Q_1(n) + Q_2(n) - \tfrac{3}{16}N_2(n) \pm \tfrac{1}{48}N_1(n),$$

for an *odd* n. In both cases $\omega(n)$ stands for

$$\omega(n) = \tfrac{5}{16}(2T(n) \mp U(n)).$$

Now it remains to substitute for $T(n)$ and $U(n)$ their expressions and to eliminate $Q_1(n)$ in case of an *even* n to get finally

$$\pm \sum (h/5)\, F(n - h^2) = -\tfrac{5}{6}\Phi(n/4) - \Psi_{\mp 1} \quad (n/4) + \tfrac{1}{2}\phi(n) - \tfrac{5}{6}Q_2(n)$$

for an *even* n and

$$\pm \sum (h/5)\, F(n-h^2) = -\tfrac{5}{8}\Phi(n) + \tfrac{1}{2}\Psi_{\mp 1}(n) + \tfrac{5}{16}(Q_1(n)+Q_2(n))$$

for an *odd* n. Now taking $4n$ instead of n in the first of these relations and introducing

$$H(n) = G(n) - F(n)$$

we find easily

$$\pm 3 \sum (h/5)\, H(4n-h^2) = \tfrac{5}{2}\Phi(n) - 3\Psi_{\mp 1}(n).$$

There is no difficulty any more to establish the following system of Gierster's relations corresponding to $n \equiv \pm 1 \pmod 5$:

$$2 \sum H(4n-25h^2) = \Phi(n) + 2\Psi(n) - 2\Psi_{\mp 1}(n)$$

$$6 \sum_{h \equiv \pm n^{1/2} \pmod 5} H(4n-h^2) = 4\Phi(n)$$

$$6 \sum_{h \equiv \pm 2n^{1/2} \pmod 5} H(4n-h^2) = -\Phi(n) + 6\Psi_{\mp 1}(n).$$

AUGUST 4, 1927.

Note on a Theorem of Bôcher.*

By Griffith C. Evans.

We prove the following theorem:

Theorem 1. *If $u(M)$ is a potential function for its gradient vector $\nabla u = (D_1 u, D_2 u)$, and if the equation*

$$\int \nabla_n u \, ds = 0 \tag{1}$$

is satisfied on almost all segments † in an open connected plane region Σ, then the function $u(M)$ has merely unnecessary discontinuities in σ, and when these are removed by changing the value of $u(M)$ at most in the points of a set of superficial measure 0, the resulting function has continuous derivatives of all orders and satisfies Laplace's equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0, \tag{2}$$

at every point in Σ.

The theorem is required for an adequate treatment of Poisson's and related equations. It derives originally from the well known theorem of Bôcher's to the effect that if $u(M)$ and its partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are continuous in Σ and if the equation

$$\int (\partial u/\partial n)\, ds = 0 \tag{3}$$

is satisfied for all circles, then $u(M)$ is harmonic in Σ. Less remotely it derives from a theorem of the author,‡ based on the ideas of generalized derivatives and potential, and thus more general than Bôcher's theorem, but proved in substantially the same manner after these ideas have been developed.

We generalize slightly the idea of potential function § in the following fashion.

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* Presented to the American Mathematical Society, Sept. (1927).

† That is, all rectangles in Σ formed from lines $x = a$, $y = b$ except possibly those which correspond to values of a and b constituting sets of zero measure.

‡ "Fundamental Points of Potential Theory," *Rice Institute Pamphlet*, Vol. 7 (1920), pp. 252-329; see also the author's recent book *Logarithmic Potential—Discontinuous Dirichlet and Neumann Problems*, New York (1927), Chapter VII.

§ Evans, *Rice Institute Pamphlet, loc. cit.*, p. 274. The curve w is of class Γ if it possesses a constant Γ such that

$$\int_w \frac{|\cos n, \overline{MP}|}{\overline{MP}}\, ds_P < \Gamma$$

independently of the position of M in Σ. For Theorem 1 only rectangles and circles are needed.

Let $u(M)$, D_1u, D_2u be three functions summable over every closed two-dimensional region in Σ, E some fixed set in Σ of superficial measure 0, and w an arbitrary regular closed curve of class Γ which contains of E no subset which has positive linear measure on w. Then if

$$(4) \qquad \begin{aligned} \int_w u\,dy &= \int_\sigma D_1u\,d\sigma \\ \int_w u\,dx &= -\int_\sigma D_2u\,d\sigma, \end{aligned}$$

σ being the region bounded by w, we say that $u(M)$ is the potential function of its generalized or vector derivative or gradient vector $\nabla u = (D_1u, D_2u)$. Any vector whose components are identical with these except on a set of superficial measure 0 is equally a gradient vector for $u(M)$.

It follows immediately that

$$\int_w u\,d\alpha' = \int_\sigma D_\alpha u\,d\sigma,$$

where

$$D_\alpha u = D_1u \cos x\alpha + D_2u \cos y\alpha$$

and α' is the direction $\pi/2$ in advance of the arbitrary direction α, and also *

$$(5) \qquad \int_w u\,d\theta = \int_\sigma (1/r)\, D_r u\,d\sigma$$

if w does not contain the pole of the system of coordinates (r, θ).

The proof of the author's theorem, already cited, applies to functions which are the potential functions of their vector derivatives in the more general sense given above, and even when (1) holds merely for almost all circles which have a common center M_0, for every M_0 in Σ. But in the theorem now to be proved the hypothesis does not require the vanishing of this total flux on a continuous or "almost continuous" family of curves about a point which are level loci of a harmonic function with an isolated singularity at the point. Hence Bôcher's device is not sufficient for the proof and an indirect method must be used.

Let then $u_\mu(M)$ be the μ-approximation †

$$u_\mu(M) = (1/4\mu^2) \int_{-\mu}^{\mu} \int_{-\mu}^{\mu} u(x+\xi, y+\eta)\,d\xi\,d\eta.$$

* Evans, *loc. cit.*, p. 282 and p. 287. For formula (23) p. 282 and that on the bottom of page 279 there should be required an additional hypothesis, as the author has remarked elsewhere. This does not affect the validity of the Theorem of Art. 5. 33, requiring merely an obvious modification of the first part of the proof.

† H. E. Bray, "Proof of a Formula for an Area," *Bulletin of the American Mathematical Society*, Vol. 29 (1923), pp. 264-270.

We know that $u_\mu(M)$ is bounded and continuous in any given closed region in Σ, μ small enough; similarly for the quantities

$$\partial u_\mu/\partial x = (1/4\mu^2) \int_{-\mu}^{\mu} \int_{-\mu}^{\mu} D_1 u(x+\xi, y+\eta)\, d\xi\, d\eta,$$

$$\partial u_\mu/\partial y = (1/4\mu^2) \int_{-\mu}^{\mu} \int_{-\mu}^{\mu} D_2 u(x+\xi, y+\eta)\, d\xi\, d\eta,$$

since $u(M)$ is a potential function of its vector derivative. In fact these statements result immediately from the properties (3), (6), (7) of Bray's paper. Incidentally, of course, u_μ is a potential function of its vector derivative.

Consider now an arbitrary circumference C in Σ, and take μ less than the distance from C to the frontier of Σ. Let s be a finite plurisegment interior to C, no point of which is distant from C by more than some δ which will eventually be arbitrarily small. We have

$$I_s = \int_s (\partial u_\mu/\partial n)\, ds = \int_s -(\partial u/\partial x)\, dy + (\partial u/\partial y)\, dx$$

$$= (1/4\mu^2) \int_{-\mu}^{\mu} d\xi \int_{-\mu}^{\mu} d\eta \int_s (-D_1 u(x+\xi, y+\eta)\, dy + D_2 u(x+\xi, y+\eta)\, dx),$$

by means of Bray's theorem (6). Hence

$$I_s = (1/4\mu^2) \int_{-\mu}^{\mu} d\xi \int_{-\mu}^{\mu} d\eta \int_{s_{\xi,\eta}} (-D_1 u(\phi, \psi)\, d\psi + D_2 u(\phi, \psi)\, d\phi),$$

where $s_{\xi,\eta}$ is the locus obtained by displacing s in the amount (ξ, η). But the inside integral vanishes for almost all (ξ, η), and, therefore,

$$I_s = 0.$$

If we compare I_s with the integral

$$I = \int_C (\partial u_\mu/\partial n)\, ds,$$

we see that the latter vanishes also. In fact I and I_s are merely Riemann integrals of continuous functions, and I_s may be made to differ by as little from I as desired. Hence by Bôcher's theorem, the function $u_\mu(M)$ is harmonic in Σ.

We return now to the function $u(M)$, by letting μ approach zero. Let σ_R be the region contained between two arbitrary concentric circles in Σ of radii r_0 and R, respectively, $r_0 < R$. We have

$$\lim \int_{\sigma_R} (1/r)\, (\partial u_\mu/\partial r)\, d\sigma = \int_{\sigma_R} (1/r) D_r u\, d\sigma$$

by Bray's theorem (8). But the integral of the left hand member vanishes, as is seen when it is expressed as an iterated integral, and therefore

$$0 = \int_{\sigma_R} (1/r) D_r u \, d\sigma = \int_{r_0}^{R} (dr/r) \int_{C_r} D_r u \, ds,$$

where C_r is the concentric circle of radius r. Hence

$$\int_{C_r} D_r u \, ds = 0, \quad \text{for almost all } r. \tag{6}$$

This result is however sufficient, as noted above, since $u(M)$ is the potential of its vector derivative, to imply that $u(M)$ has merely removable discontinuities at a set of points of zero measure, and, when these are removed, becomes harmonic.

In a similar fashion we may prove a more general theorem, useful also in the treatment of equations of elliptic type. We shall say that a *translatable* regular net consists of the curves in Σ obtained by arbitrary parallel displacements of a regular net *; and an equation in functions of curves will be said to hold almost everywhere on such a net if, given a cell w, it holds for almost all the homologous cells obtained by displacing it parallel to itself in the x and y directions.† We may state then the theorem:

THEOREM II. *Let $u(M)$ be a potential function for its vector derivative $(D_1 u, D_2 u) = \nabla u$ in Σ, and let the equation*

$$\int_w \nabla_n u \, ds = 0$$

be satisfied almost everywhere on a translatable regular net of class Γ in Σ. Then the conclusion of Theorem I is valid.

It is hardly necessary to say that for theorems of this sort the case $n = 2$ is typical of n dimensions.‡ Some five years ago Professor Bray and the author did however work over the corresponding part of the article on "Fundamental Points of Potential Theory" for the case $n = 3$, without finding any essential novelty; the set corresponding to E' of Art. 5. 32 of that paper is perhaps a little more difficult to define. Lately, among other theorems, Professor Raynor has given Bôcher's theorem for three dimensions.§

* H. E. Bray and G. C. Evans, "A Class of Functions Harmonic in a Sphere," *American Journal of Mathematics*, Vol. 49 (1927), pp. 153-180; see p. 169. Here the regular net may be regarded as given over the entire plane, and the number of cells in any lattice containing points of a bounded region will be finite.

† Or, instead of the group of translations T we may for instance use throughout the group of transformations $S^{-1}TS$ where S is a fixed conformal transformation—this of course only in the plane.

‡ See the *Cambridge Colloquium Lectures*, Part I, New York (1918), p. 77, and G. Bouligand, "Fonctions harmoniques. Principes de Picard et de Dirichlet," *Mémorial des sciences mathématiques*, fasc. XI, Paris (1926), pp. 7-9.

§ *Bulletin of the American Mathematical Society*, Vol. 52 (1926), pp. 654-658.

Generalized Neumann Problems for the Sphere.*

By Griffith C. Evans.

Introduction. Let $v(M)$ be harmonic within a sphere S of unit radius, and let w_r be a simple closed regular curve, with interior region ω_r, on the concentric sphere S_r of radius $r < 1$, whose projection on S is w. We consider the problem of determining $v(M)$ within S in such a way that the quantity

$$\int_{\omega_r} (\lambda v + \partial v/\partial r)\, d\sigma$$

takes on given values $H(w)$ as r approaches 1, $H(w)$ being a bounded additive function of regular curves on S, with regular discontinuities.

For the class of functions for which $\int | \partial v/\partial r |\, d\sigma$, extended over the sphere S_r, remains bounded as r approaches 1, we shall show that there is a unique solution of the problem provided that λ is not one of a set of special parameter values, this set being the collection of negative integers and zero. For the value $\lambda = 0$ there is a solution, provided $H(S) = 0$, unique except for an additive constant; in this last case if $H(w)$ is absolutely continuous the problem reduces essentially to the Neumann problem with boundary values summable in the Lebesgue sense.

We shall write $v(M)$ as the potential of a simple distribution of mass and obtain the mass function in terms of $H(w)$ by solving a Stieltjes integral equation. In a supplementary note, we discuss briefly the situation in n dimensions, with special reference to the irregular case, $n = 2$.

1. Consider first the condition that

$$\int_{S_r} | \partial v/\partial r + \lambda v |\, d\sigma$$

remains bounded as r approaches 1, and let

$$\partial v/\partial r + \lambda v = \xi(M).$$

We have

$$v(r, P) = e^{-\lambda r} \int_{r_0}^{r} \xi(r, P)\, e^{\lambda r} dr + e^{\lambda(r_0 - r)} v(r_0, P),$$

* Preliminary report presented to the American Mathematical Society, April, 1927.

where v is given in terms of r and the radial projection P of M on S. Hence

$$\int_{S_r} |v(M)|\, d\sigma = r^2 \int_S |v(r,P)|\, d\sigma_P$$

$$\leq e^{-\lambda r} \int_{r_0}^{r} dr' e^{\lambda r'} \int_S |\xi(r',P)|\, d\sigma_P + e^{\lambda(r_0-r)} \int_S |v(r_0,P)|\, d\sigma_P,$$

which is bounded, from the given condition. But this implies, even when λ is 0, that $\int_{S_r} |v(M)|\, d\sigma$ is bounded, $r<1$.

It is evident then that *the two conditions*

$$\text{(j)} \qquad \int_{S_r} |\partial v/\partial r + \lambda v|\, d\sigma < N_1, \quad \int_{S_r} |\partial v/\partial r|\, d\sigma < N, \quad r<1,$$

are equivalent, if the functions v, $\partial v/\partial r$ are continuous within S.

2. Let us assume that (j) holds. Then both the quantities

$$\int_{S_r} |r\partial v/\partial r|\, d\sigma, \qquad \int_{S_r} |r\partial v/\partial r + v/2|\, d\sigma$$

remain bounded within S; and since v, r and $r\partial v/\partial r + v/2$ are both harmonic, $r<1$, we have *

$$\text{(1)} \qquad r\frac{\partial v}{\partial r} = \frac{1}{4\pi}\int_S \frac{(1-r^2)}{(1+r^2-2r\cos\alpha)^{3/2}}\, dG(s_P)$$

$$\text{(2)} \qquad r\frac{\partial v}{\partial r} + \frac{v}{2} = \frac{1}{4\pi}\int_S \frac{(1-r^2)}{(1+r^2-2r\cos\alpha)^{3/2}}\, dF(s_P)$$

where $F(w)$, $G(w)$ are bounded additive functions of regular curves on S (in particular, of segments s), with regular discontinuities; and also

$$\text{(2')} \qquad \lim_{r=1}\int_{\omega_r} r\frac{\partial v}{\partial r}\, d\sigma = G(w), \quad \lim_{r=1}\int_{\omega_r}\left(r\frac{\partial v}{\partial r} + \frac{v}{2}\right) d\sigma = F(w),$$

$$v(0) = \frac{F(S)}{2\pi}$$

In these formulae, α is the arc PQ on the unit sphere, Q being the projection of M, and $MP^2 = 1 + r^2 - 2r\cos\alpha$.

Consider now the quantity, which is also harmonic within S:

$$X(M) = \frac{1}{2\pi}\int_S \frac{dF(s)}{(1+r^2-2r\cos\alpha)^{\frac{1}{2}}}$$

* Bray and Evans, "A Class of Functions Harmonic within the Sphere," *American Journal of Mathematics*, Vol. 49 (1927), pp. 153-180.

An obvious calculation yields the result

$$r\partial X/\partial r + X/2 = r\partial v/\partial r + v/2$$
$$r^{1/2}X = r^{1/2}v + \psi(\theta, \phi)$$

where $\psi(\theta, \phi)$ is a continuous function of the coordinates θ, ϕ on the sphere. Hence, since the left hand member of the above equation vanishes when $r = 0$, we have $\psi(\theta, \phi) = 0$, and the functions X and v are identical. Hence

$$(3) \qquad v(M) = \frac{1}{2\pi} \int_S \frac{dF(s)}{(1 + r^2 - 2r\cos\alpha)^{1/2}} .$$

From (2) we have

$$G(w) = F(w) - \lim_{r=1} \frac{1}{4\pi} \int_S dF(s') \int_{\omega_r} \frac{d\sigma}{(1 + r^2 - 2r\cos\alpha)^{1/2}}$$
$$= F(w) - \frac{1}{4\pi} \int_S dF(s') \int_\omega \frac{d\sigma}{[2(1 - \cos\alpha)]^{1/2}}$$

on account of the uniform convergence of the improper integral. Hence

$$(4) \qquad F(w) = G(w) - \frac{1}{8\pi} \int_S dF(s') \int_\omega \frac{d\sigma}{\sin\alpha/2} .$$

By the same sort of calculation we have also

$$(5) \qquad F(w) = H(w) + \frac{1 - 2\lambda}{8\pi} \int_S dF(s') \int_\omega \frac{d\sigma}{\sin\alpha/2}$$

where, in virtue of the condition (j),

$$(6) \qquad r\frac{\partial v}{\partial r} + \lambda v = \frac{1}{4\pi} \int_S \frac{(1 - r^2)\, dH(s)}{(1 + r^2 - 2r\cos\alpha)^{3/2}}$$

$$(6') \qquad \lim_{r=1} \int_{\omega_r} (r\partial v/\partial r + \lambda v)\, d\sigma = \lim_{r=1} \int_{\omega_r} (\partial v/\partial r + \lambda v)\, d\sigma = H(w),$$

the function $H(w)$ being again a bounded additive function of regular curves, with regular discontinuities. Accordingly we state the following theorem.

THEOREM 1. *That $v(M)$ be harmonic within the sphere and that also* (j) *hold is a necessary and sufficient condition that $v(M)$ be given by a formula* (3), *in which $F(w)$ is a bounded additive function of regular curves, with regular discontinuities. In this case the $G(w)$, defined by* (2'), *and the $H(w)$, defined by* (6'), *are functions of the same sort, and the equations* (1), (2), (4), (5), (6) *are all valid.*

We have just proved the sufficiency of the condition. In order to estab-

lish the necessity, we notice that $v(M)$, given by (3), is harmonic, and in particular that $\int_{S_r} \partial v/\partial r \, d\sigma = 0$. Moreover (2) is satisfied, whence

$$\int_{S_r} | \partial v/\partial r + v/2 | \, d\sigma$$

is bounded,* and (j) is established by § 1.

COROLLARY. If $v(M)$ is given by a formula (3) in terms of an $F_1(s)$ which is a bounded additive function of plurisegments, its discontinuities not of necessity being regular, then also

$$v(M) = \frac{1}{2\pi} \int_S \frac{dF(s)}{(1 + r^2 - 2r\cos\alpha)^{1/2}} = \frac{1}{2\pi} \int_S \frac{dF(w)}{(1 + r^2 - 2r\cos\alpha)^{1/2}}$$

where

$$F(w) = \int_S q(P, w) \, dF_1(s)$$

$q(P, w)$ being the circular density of ω on S at P, a function measurable in the Borel sense.†

3. We pass now to the integral equation (5). If it has a solution $F(w)$, the function $v(M)$, given by (3), will be harmonic within the sphere, of class (j) and will satisfy the boundary condition (6′). On the other hand, if $v(M)$, of class (j), is harmonic within the sphere it will be given by a formula (3), where $F(w)$ is determined by (2′); and if also it satisfies the boundary condition (6′), the $F(w)$ will satisfy (5).

Since the discontinuities of $H(w)$ are regular, it is sufficient to consider the segments s on the sphere. Write

$$F(s) = H(s) + \Phi(s).$$

Equation (5) becomes

$$\Phi(s) = \frac{1 - 2\lambda}{8\pi} \int_S dH(s') \int_\sigma \frac{d\sigma}{\sin(\alpha/2)} + \frac{1 - 2\lambda}{8\pi} \int_S d\Phi(s') \int_\sigma \frac{d\sigma}{\sin(\alpha/2)}$$

where the first term of the right hand member is to be regarded as a known function, and σ is the region bounded by s.

We shall show first that on the hypothesis that $H(s)$, $\Phi(s)$ are bounded additive functions of plurisegments, the function $\Phi(s)$ will be *absolutely continuous.* In fact, if we write

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* Bray and Evans, *loc. cit.*, p. 168.

† Bray and Evans, *loc. cit.*, p. 168.

$$p_n(\alpha) = \frac{1}{\sin(\alpha/2)}, \qquad \frac{1}{\sin(\alpha/2)} < n$$
$$= n, \qquad \frac{1}{\sin(\alpha/2)} \geq n,$$

the function so defined will be not negative, continuous on the sphere and non-decreasing with n, and

$$\int_S dH(s') \int_\sigma p_n(\alpha)\, d\sigma = \int_\sigma d\sigma \int_S p_n(\alpha)\, dH(s').$$

Moreover

$$\left| \int_S dH(s') \int_\sigma p_n(\alpha)\, d\sigma \right| \leq \int_S dT(s') \int_\sigma p_n(\alpha)\, d\sigma,$$

where $T(s)$ is the total variation function for $H(s)$. But this is

$$\leq \int_\sigma dT(s') \int_\sigma \frac{d\sigma}{\sin(\alpha/2)}.$$

Hence the quantity

$$\int_S \frac{dH(s')}{\sin(\alpha/2)}$$

exists almost everywhere on S, and represents a summable function;* moreover:

$$(7) \qquad \int_\sigma d\sigma \int_S \frac{dH(s')}{\sin(\alpha/2)} = \int_\sigma dH(s') \int_\sigma \frac{d\sigma}{\sin(\alpha/2)}$$

But the left hand member is an absolutely continuous function, and the point is proved.

If then we write

$$h(P) = \frac{1-2\lambda}{8\pi} \int_S \frac{dH(s')}{\sin(\alpha/2)}, \quad f(P) = \Phi'(\sigma; P) = \frac{d\Phi}{d\sigma}$$

equation (5) implies the following:

$$(8) \qquad f(Q) = h(Q) + \frac{1-2\lambda}{8\pi} \int_S \frac{f(P)}{\sin(\alpha/2)}\, d\sigma_P$$

$$(8') \qquad \Phi(s) = \int_\sigma f(P)\, d\sigma$$

$$F(s) = \Phi(s) + H(s).$$

On the other hand, if (8) has a solution $f(P)$, summable in the Lebesgue

* P. J. Daniell, "A General Form of Integral," *Annals of Mathematics*, Vol. 19 (1918), pp. 279-294, and continued in later papers; for the interchange of order of integration, G. C. Evans, "Fundamental Points of Potential Theory," *Rice Institute Pamphlet*, Vol. 7 (1920), pp. 252-329, see p. 258 and p. 262.

sense on S, equation (8′) defines a function $F(s)$, bounded and additive, with regular discontinuities, which is a solution of (5).

In passing, we note the particular value $\lambda = 1/2$, for which the only solution of (8) is $f(P) = 0$. For this value we have $F(s) = H(s)$ as the only solution of our problem. It is this case, as we shall see later, which corresponds to the value $\lambda = 0$ in the two dimensional problem; and thus it happens that in two dimensions the solution of the Neumann problem is given by a direct relation, while in three dimensions the "same" problem ($\lambda = 0$) is answered in terms of the integral equation (4).

The kernel of (8)

$$K(Q,P) = K(P,Q) = 1/8\pi \cdot 1/\sin(\alpha/2)$$

becomes infinite when $P = Q$; but, as we have already seen, the function $\int_S K(Q,P)\, g(P)\, d\sigma_P$ is summable if $g(P)$ is summable, since this is merely a particular case of the result established earlier in this section. Hence the customary analysis applies,* and the solutions of (8) are identical with those of the following equation:

$$(9) \qquad f(Q) = h(Q) + \nu \int_S K(Q,P)h(P)d\sigma_P + \nu^2 \int_S K_1(Q,P)h(P)d\sigma_P + \nu^3 \int_S K_2(Q,P)f(P)d\sigma_P,$$

unless $\nu = 1 - 2\lambda$ is such that ν^3 is a root of the twice iterated kernel

$$K_2(Q,P) = \int_S K(Q,X)\, K(X,Y)\, K(Y,P)\, d\sigma_X d\sigma_Y$$

which is easily shown to be a continuous function. The solution of equation (8) and therefore of equation (5) is accordingly unique if λ does not correspond to one of these exceptional values of ν. These values of ν are known to be the positive odd integers,† to which correspond, as values of λ, the negative integers and zero. But we may find them again, and at the same time completely solve our problem by means of a simple differential equation of the first order.

4. We return then to the equation (6). To say that (5) has one or more solutions implies that the equation (6) may be solved for $v(M)$ and that the resulting function $v(M)$ will be harmonic, in particular at the point $r = 0$. Conversely if $v(M)$ satisfies (6) it will be of class (j), and the corresponding $F(w)$ will therefore satisfy (5). If we write

* Goursat, *Cours d'Analyse*, Vol. 3, Paris (1915), p. 355 and p. 382.

† Goursat, *loc. cit.*, p. 513.

$$h(M) = h(r, P) = \frac{1}{4\pi} \int_S \frac{(1 - r^2)\, dH(s)}{(1 + r^2 - 2r\cos\alpha)^{3/2}}$$

the equation to be solved may be written in the form

$$r\partial v/\partial r + \lambda v = h(r, P), \tag{10}$$

where $h(r, P)$ is harmonic inside S, and may therefore be expanded in the form

$$h(r, P) = Y_0 + \sum_1^\infty Y_k(P) r^k, \tag{10'}$$

where $Y_k(P)$ is a surface or spherical harmonic of degree k, the series being convergent, uniformly in P, for any value of $r < 1$.

Equation (10) may be written in the form

$$\partial(r^\lambda v)/\partial r = Y_0 r^{\lambda-1} + \sum_1^\infty Y_k(P) r^{k+\lambda-1}.$$

Hence if λ is not zero or a negative integer, we have

$$r^\lambda v = (1/\lambda) Y_0 r^\lambda + \sum_1^\infty [Y_k(P)/(k + \lambda)] r^{k+\lambda} + C(P)$$

and therefore

$$v = (1/\lambda) Y_0 + \sum_1^\infty [Y_k(P)/(k + \lambda)] r^k \tag{11}$$

since $C(P) r^{-\lambda}$ cannot be harmonic at the origin unless $C(P) = 0$.

If λ is a negative integer or zero, the expansion yields the following:

$$r^\lambda v = \sum_0^{\infty\prime} [Y_k(P)/(k + \lambda)] r^{k+\lambda} + Y_{-\lambda}(P) \log r + C(P),$$

where in the summation the term $k = -\lambda$ is lacking. Hence

$$v(M) = \sum_0^{\infty\prime} [Y_k(P)/(k + \lambda)] r^{k+\lambda} + Y_{-\lambda}(P) r^{-\lambda} \log r + C(P) r^{-\lambda}. \tag{11'}$$

But this will not be harmonic in the neighborhood of the origin unless $Y_{-\lambda}(P) = 0$. Conversely, if this condition is satisfied, $v(M)$ will be a solution, in which $C(P)$ is an arbitrary surface harmonic of degree $\mu = -\lambda$.

The condition $Y_\mu(P) = 0$ may be written in the form

$$\int_S Y_\mu(P) h(r, P)\, d\sigma_P = 0,$$

as is seen from (10'). But

$$\int_S Y_\mu(P)h(r,P)\,d\sigma_P = \frac{1}{4\pi}\int_S dH(s_{P'})\int_S \frac{Y_\mu(P)(1-r^2)}{(1+r^2-2r\cos PP')^{3/2}}\,d\sigma_P$$

$$= (r^\mu/4\pi)\int_S Y_\mu(P)\,dH(s_P).$$

Hence we have the following theorem.

THEOREM 2. *If λ is not a negative integer or zero there is one and only one function $v(M)$ harmonic inside the sphere and of class* (j), *such that*

$$\lim_{r=1}\int_\omega \{\lambda v(r,P) + \partial v(r,P)/\partial r\}\,d\sigma = H(w), \tag{6'}$$

$H(w)$ being an arbitrary, bounded, additive function of regular curves, with regular discontinuties.

If $\lambda = -\mu$ is a negative integer there is no solution of the class (j) *subject to* (6') *unless $H(w)$ satisfies the $2\mu+1$ independent conditions*

$$\int_S Y_\mu(P)\,dH(s_P) = 0, \tag{12}$$

$Y_\mu(P)$ being the general surface harmonic of degree μ. If (12) *is satisfied the solution exists, is of class* (j) *and is determined except for the addition of an arbitrary solid harmonic $Y_\mu(P)\,r^\mu$* of degree μ. In this case $F(w)$ is *determined as a solution of* (5) *except for an additive term of the form $\int_\omega Y_\mu(P)\,d\sigma_P$.*

If $\lambda = 0$, the necessary condition is $H(S) = 0$, and the function $v(M)$ *is determined except for an additive constant. In this case $F(w)$ is determined, as a solution of* (5), *except for an arbitrary additive term of the form $C\omega$.*

5. From this point results may be obtained corresponding to those of Bray and Evans in the article already cited. It is unnecessary to repeat them in detail except to point out that

$$\lim_{r=1}\,(r\partial v/\partial r + \lambda v) = dH/d\sigma$$

wherever that derivative exists, that is, almost everywhere on S; and that therefore

$$\lim_{r=1}\partial v/\partial r, \quad \lim_{r=1} v$$

exist for almost all points of S, since the condition (j) implies the condition

(i) $$\int_{S_r} |v(M)|\, d\sigma \text{ is bounded.}$$

In fact these limits all exist whenever $\partial G/\partial\sigma$ exists. For if lim $\partial v/\partial r$ exists, then along that radius $\partial v/\partial r$ is bounded, and lim v exists.

Finally, *if we replace the condition* (j) *by the condition*

(jj) $\int_{\omega_r} (\partial v/\partial r)\, d\sigma$ is absolutely continuous, uniformly in r,

a condition which is satisfied, in particular, if $\partial v/\partial r$ *is bounded within* S, *and the condition* (6′) *by the condition*

$$(6'') \qquad \lim_{r=1} (\partial v/\partial r + \lambda v) = h(P), \quad h(P) \text{ summable on } S,$$

the Theorem 2 remains valid. In this case $H(s) = \int_\sigma h(P)\, d\sigma$.

Thus we have a solution of the Neumann problem itself, and of a class of problems related to it.

6. *Supplementary note on the problem in n-dimensions.* The case of three dimensions being typical, we content ourselves with a brief description of the formulae involved in the n-dimensional situation. The Poisson integral for the hypersphere S of radius R takes the form

$$(13) \qquad u(M) = u(r, Q) = \frac{1}{RA_n} \int_S \frac{R^2 - r^2}{\overline{MP}^n}\, dF(s),$$

where $A_n = 2\pi^{n/2}/\Gamma(n/2)$ is the $(n-1)$-dimensional "area" or measure of the hypersphere of unit radius. Its n-dimensional measure of "volume" is of course $2\pi^{n/2}/[n\Gamma(n/2)]$. In (13) we have written

$$(14) \qquad F(s) = \lim_{r=R} F(r, s_r) = \lim_{r=R} \int_{\sigma_r} u(M)\, d\sigma_r,$$

the $d\sigma_r$ being $r^{n-1}\, d\sigma$, with $d\sigma$ as the element of area of the unit hypersphere; in fact the constant before the integral has been chosen so that

$$(14') \qquad u(0) = F(S)/A_n(R) = F(S)/R^{n-1}A_n$$

For the corresponding potential of a single layer we write

$$(15) \qquad v(M) = \frac{2}{(n-2)A_nR} \int_S \frac{dF(s)}{\overline{MP}^{n-2}} =$$

$$\frac{2}{(n-2)A_nR} \int_S \frac{dF(s)}{(R^2 + r^2 - 2Rr\cos\alpha)^{(n/2)-1}}$$

We have

$$r\,\frac{\partial v}{\partial r} = -\frac{1}{A_n R}\int_S \frac{2r^2 - 2Rr\cos\alpha}{(R^2+r^2-2Rr\cos\alpha)^{n/2}}\,dF(s),$$

and therefore,

$$r\partial v/\partial r + [(n-2)/2]v = u$$

$$r\partial v/\partial r + \lambda v = u + [\lambda - (n-2)/2]v = [(u - (n-2-2\lambda/2]v$$

$$= u - \frac{n-2-2\lambda}{(n-2)RA_n}\int_S \frac{dF(s)}{\overline{MP}^{n-2}}\ .$$

Hence we write

$$\lim_{r=R}\int_{S_r} r(\partial v/\partial r)\,d\sigma_r = G(s) = R\lim_{r=R}\int_{S_r}(\partial v/\partial r)\,d\sigma_r$$

$$\lim_{r=R}\int_{S_r}(r\partial v/\partial r + \lambda v)\,d\sigma_r = H(s),$$

where $G(s)$ and $H(s)$ are bounded additive functions of $(n-2)$-dimensional plurisegments on the hypersphere,* provided the condition (j) is valid:

(j) $$\int_{S_r} |\,\partial v/\partial r\,|\,d\sigma_r \text{ is bounded, } r < R.$$

We have then the integral equations relating $H(s)$, $G(s)$ and $F(s)$, to determine $F(s)$:

(16) $$G(s) = F(s) - \frac{1}{RA_n}\int_S dF(s_P)\int_\sigma \frac{d\sigma}{\overline{QP}^{n-2}}$$

(17) $$H(s) = F(s) - \frac{n-2-2\lambda}{(n-2)RA_n}\int_S dF(s_P)\int_\sigma \frac{d\sigma}{\overline{QP}^{n-2}}$$

Here $G(s)$ of course vanishes, if $v(M)$ is given, and for particular values of $\lambda \neq 0$ there will also be necessary conditions on the $H(s)$.

In order to find these conditions we have merely to repeat the analysis of § 4. The characteristic values of λ will again be the negative integers $-\mu$ and 0; and if we write

$$h(r,Q) = \frac{1}{RA_n}\int_S \frac{(R^2-r^2)\,dH(s)}{\overline{MP}^n} = \sum_0^\infty X_k(Q)r^k$$

the functions $X_k(Q)$ being orthogonal, the condition to be satisfied by $H(s)$ will be

$$\int_S Y_\mu(P)\,dH(s_P) = 0,$$

* Edmonson, *Poisson's Integral and Plurisegments on the Hypersphere*, in preparation.

where $Y_\mu(P)r^\mu$ is the general homogeneous polynomial of degree μ, which satisfies Laplace's equation.

The case $n = 2$ is not strictly regular, as is seen from the above formulae. But the problem corresponding to $\lambda = 0$ has already been considered in detail.* We write

$$v(M) = (1/\pi R)\int_S \log(1/\overline{MP})\, dF(s) + A, \text{ with } F(S) = 0, \quad A = v(0),$$

$$u(M) = (1/2\pi R)\int_S [(R^2 - n^2)/\overline{MP^2}]\, dF(s),$$

and therefore

$$r\partial v/\partial r = u$$

$$r(\partial v/\partial r) + \lambda v = u + (\lambda/\pi R)\int_S \log(1/MP)\, dF(s) + \lambda A;$$

and the two equations corresponding to the integral equations (16), (17) are

$$G(s) = F(s)$$

$$H(s) = F(s) + (\lambda/\pi R)\int_S dF(s_P)\int_S \log(1/QP)\, ds_Q + \lambda A s, \tag{18}$$

where, when $\lambda \neq 0$, we may still keep $F(S) = 0$ by taking $A = v(O) = H(S)/\lambda S$. The second of the equations (18) may therefore be written in the form

$$H(s) - [H(S)/\lambda S] = F(s) + (\lambda/\pi R)\int_S dF(s_P)\int_S \log(1/\overline{QP})\, ds_Q.$$

From this point on, the analysis offers nothing novel.

The conditions may be stated in a form invariant of conformal transformation. And if this is done the results will apply to general [simply connected] regions, and the special parameter values of λ will be unaltered. To this end we let $b(M_0, M)$ be the function conjugate to the Green's function $g(M_0, M)$ and write the condition (j) in the form

$$\int_{g=\text{const.}} |\partial v/\partial g|\, db = \int_{g=\text{const.}} |\partial v/\partial n|\, ds < N.$$

The boundary condition (6′) takes the form

$$\lim_{g=0}\int_{b'}^{b''} (v + \lambda\partial v/\partial g)\, db = H(b', b'') = H(s).$$

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* Evans, *The Logarithmic Potential—Discontinuous Dirichlet and Neumann Problems*, New York (1927), Ch. IV.

Similarly in the condition (jj) the absolute continuity of the integral

$$\int_{b'}^{b} (\partial v/\partial g)\, db = \int_{(b')}^{(b'')} (\partial v/\partial n)\, ds$$

extended along curves $g = \text{const.}$, is uniform, $g > 0$; and the condition (6″) takes the form

$$\lim_{g=0} (v + \lambda \partial v/\partial g) = h(b),$$

$h(b)$ being summable with respect to b.

*Under these conditions there is a unique solution of the respective boundary value problems, unless $\lambda = 0$ or a negative integer.**

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* See the treatment for $\lambda = 0$ in Evans, *loc. cit.*, Ch. V.

On Taylor's Series Admitting the Circle of Convergence as a Singular Curve.*

By J. J. Gergen and D. V. Widder.†

1. *Introduction.* It is a familiar fact that certain Taylor's series

$$\sum_{n=0}^{\infty} a_n x^n \tag{1}$$

define functions which can not be continued analytically beyond their circles of convergence. The series of Weierstrass and Fredholm,

$$\sum_{n=0}^{\infty} x^{2^n}, \tag{2}$$

$$\sum_{n=0}^{\infty} x^{n^2}, \tag{3}$$

are classic examples. Both of these series are *lacunary;* that is, they are particular cases of (1) in which certain of the coefficients are zero. General classes of such series have been studied by a great number of authors including J. Hadamard, E. Borel, S. Mandelbrojt, G. Faber, and E. Fabry. We recall several of the results. It was shown by Hadamard that the series

$$\sum_{m=0}^{\infty} a_m x^m = \sum_{n=0}^{\infty} a_{\lambda_n} x^{\lambda_n}, \tag{4}$$

where

$$a_m = 0 \text{ when } m \neq \lambda_n, \quad a_{\lambda_n} \neq 0,$$

admits the circle of convergence as a cut if

$$(\lambda_{n+1} - \lambda_n)/\lambda_n$$

is greater than some positive constant for all n. The series (2) of Weierstrass is a case in point. More generally, it has been shown ‡ that the conclusion is the same if

$$(\lambda_{n+1} - \lambda_n)/\lambda^{\sigma}{}_n, \qquad 0 < \sigma \leq 1,$$

* Preliminary report presented to the American Mathematical Society, September 9, 1927. A preliminary account of the results of this paper was published in a note in *Comptes Rendus*, Vol. 185 (1927), pp. 829-831. Since the publication of that note, the authors have simplified several of the proofs.

† National Research Fellow.

‡ For complete references see J. Hadamard and S. Mandelbrojt, "La Série de Taylor et son prolongement analytique," *Collection Scientia*, Gauthier-Villars, 1926.

is greater than a positive constant for all n. This was first proved for the case $\sigma = \frac{1}{2}$ by Borel. It is seen that this criterion applies to the series (3) of Fredholm.

In a recent paper Mandelbrojt * has developed a new type of criterion. He has shown that if ϕ is any irrational number, the series (4) admits its circle of convergence as a cut if $e^{2\pi i\lambda_n\phi}$ has a limit as n becomes infinite, and if a_{λ_n} has a limit different from zero. The proof of the theorem is based on a theorem of Cesàro. In the present paper the authors prove a similar theorem, replacing the condition that a_{λ_n} have a limit by the restriction that they all lie in a single quadrant of the complex plane. The method of proof is simplified, no use of Cesàro's theorem being necessary. By means of the criterion thus developed the authors are able to give a new proof of the theorem of Hadamard and its generalizations mentioned above. It has been the aim to make the proof elementary in character. It will be seen by an examination of the following sections that the proof is based on only two theorems from the theory of analytic continuation, two theorems that lie at the very foundation of that theory.

2. *A Lemma.* The proof of the theorem of the next section is made by use of the following

LEMMA. *The function defined by the series*

$$(5) \qquad f(x) = \sum_{n=0}^{\infty} a_{\lambda_n} x^{\lambda_n}, \qquad \overline{\lim_{n=\infty}} \mid a_{\lambda_n} \mid^{1/\lambda_n} = 1,$$

has the point $x = 1$ as a singular point if the real part a_{λ_n}' of a_{λ_n} satisfies the conditions $a_{\lambda_n}' > 0$, $\overline{\lim_{n=\infty}}\ a_{\lambda_n}'^{\,1/\lambda_n} = 1$.

This Lemma is a generalization of the well-known fact that the function defined by (5) has the point $x = 1$ singular if the a_{λ_n} are all real and positive.† Indeed, this result enables us to see at once that the function

$$\phi(x) = \sum_{n=0}^{\infty} a_{\lambda_n}' x^{\lambda_n}$$

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* *Comptes Rendus*, Vol. 184 (1927), pp. 1307-1309. It may be shown that the set of functions which satisfies the conditions of Mandelbrojt and the set which satisfies the conditions of the theorem of this paper form subclasses of the class for which $\lim (\lambda_{n+1} - \lambda_n) = \infty$. It is well known that a function of this latter class has its circle of convergence as a singular curve. [Gergen.]

† Compare a generalization by P. Dienes, "Essai sur les singularités des fonctions analytiques," *Journal de Mathématiques*, 6 série, t. 5 (1909), p. 338.

has the point $x = 1$ singular. That is, if β is any real number between zero and unity, the series

$$\sum_{n=0}^{\infty} \frac{\phi^{(n)}(\beta)}{n!} (x-\beta)^n$$

has the radius of convergence equal to $1-\beta$. Hence for every positive ϵ there exist an infinite number of integers m for which

$$\frac{\phi^{(m)}(\beta)}{m!} = \sum_{p=0}^{\infty} \frac{(m+p)!}{m!\,p!} a_{m+p}' \beta^p > \left(\frac{1-\epsilon}{1-\beta}\right)^m. \tag{6}$$

We wish to prove that $x = 1$ is a singular point of (5), or that

$$\left| \frac{f^{(m)}(\beta)}{m!} \right| > \left(\frac{1-\epsilon}{1-\beta}\right)^m \tag{7}$$

for an infinite number of integers m. Set

$$a_{\lambda_n} = a_{\lambda_n}' + i\, a_{\lambda_n}'', \quad \psi(x) = \sum_{n=0}^{\infty} a_{\lambda_n}'' x^{\lambda_n}.$$

Then

$$f(x) = \phi(x) + i\,\psi(x),$$

and

$$f^{(m)}(\beta) = \phi^{(m)}(\beta) + i\,\psi^{(m)}(\beta).$$

Since β is real, the quantities $\phi^{(m)}(\beta)$ and $\psi^{(m)}(\beta)$ are real. Hence

$$| f^{(m)}(\beta) | \geq \phi^{(m)}(\beta),$$

and (7) becomes an immediate consequence of (6). The Lemma is thus completely proved.*

3. *The Theorem. If* $\{\lambda_n\}$ *is a sequence of positive integers such that* $\lim_{n=\infty} e^{2\pi i \lambda_n \phi}$ *exists for some irrational number* ϕ, *and if* $\{a_{\lambda_n}\}$, $a_{\lambda_n} \neq 0$, *is a sequence of numbers all lying in a single quadrant of the complex plane,*† *then the function defined by the series*

$$\sum_{n=0}^{\infty} a_{\lambda_n} x^{\lambda_n}, \qquad \overline{\lim_{n=\infty}} \, | a_{\lambda_n} |^{1/\lambda_n} = 1, \tag{8}$$

admits the circle of convergence as a singular curve.

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* Note that it is not true, in general, that all the singularities of $\Sigma a_n' x^n$ are singularities of $\Sigma a_n x^n$. Witness the example

$$\Sigma a_n x^n = x/(x^2+1) + i/(x^2+1) = 1/(x-i).$$

† We agree that a_{λ_n} lies in the kth quadrant $(k = 1, 2, 3, 4)$ if

$$a_{\lambda_n} = r_{\lambda_n} \exp i\theta_{\lambda_n}, \quad (k-1)\pi/2 \leq \theta_{\lambda_n} < k\pi/2.$$

Without loss of generality we may suppose that all the a_{λ_n} lie in the first quadrant, for the case in which they lie in the kth quadrant may be reduced to this case by multiplying each term of the series (8) by $e^{-\pi i(k-1)/2}$. Obviously this has no effect on the position of the singularities of (8). Set

$$a_{\lambda_n} = a_{\lambda_n}' + i\, a_{\lambda_n}'', \quad a_{\lambda_n}' > 0, \quad a_{\lambda_n}'' \geq 0.$$

Then at least one of the following equations holds:

$$\overline{\lim_{n=\infty}}\ a_{\lambda_n}'^{\,1/\lambda_n} = 1, \qquad \overline{\lim_{n=\infty}}\ a_{\lambda_n}''^{\,1/\lambda_n} = 1.$$

For definiteness suppose the first to be true, and consider the series

$$\sum_{n=0}^{\infty} a_{\lambda_n} e^{2\pi i \lambda_n \phi_p} x^{\lambda_n}, \tag{9}$$

where p is any positive integer. We show that $x = 1$ is a singular point of this series by showing it to be a singular point of the following series (differing from it only by a constant factor):

$$-i \sum_{n=0}^{\infty} a_{\lambda_n} e^{2\pi i \psi_n} x^{\lambda_n}, \tag{10}$$

where

$$\psi_n = \lambda_n p\phi + 1/8 - p\psi,$$

$$e^{2\pi i \psi} = \lim_{n=\infty} e^{2\pi i \lambda_n \phi}.$$

Then we have

$$\lim_{n=\infty} e^{2\pi i \psi_n} = e^{\pi i/4}.$$

If we decompose the coefficients of (9) into real and imaginary parts, we obtain

$$\begin{aligned} -i\, a_{\lambda_n} e^{2\pi i \psi_n} = &-i(a'_{\lambda_n} \cos 2\pi\psi_n - a_{\lambda_n}'' \sin 2\pi\psi_n) \\ &+ (a_{\lambda_n}'' \cos 2\pi\psi_n + a_{\lambda_n}' \sin 2\pi\psi_n). \end{aligned}$$

We can now show that the coefficients of (10) satisfy all the conditions of the Lemma (after rejecting, if necessary, a finite number of terms at the beginning of the series), since

$$\lim_{n=\infty} \sin 2\pi\psi_n = \lim_{n=\infty} \cos 2\pi\psi_n = 1/2^{1/2},$$

it follows that

$$c_{\lambda_n} = a_{\lambda_n}'' \cos 2\pi\psi_n + a_{\lambda_n}' \sin 2\pi\psi_n$$

is positive for all n sufficiently large ($n > m$). Moreover,

$$\overline{\lim_{n=\infty}}\ c_{\lambda_n}^{\,1/\lambda_n} = 1, \quad n > m.$$

For,

$$\overline{\lim_{n=\infty}} (a_{\lambda_n}' \sin 2\pi\psi_n)^{1/\lambda_n} = \overline{\lim_{n=\infty}} a_{\lambda_n}'^{1/\lambda_n} = 1,$$

and

$$c_{\lambda_n} \geq a_{\lambda_n}' \sin 2\pi\psi_n, \quad n > m.$$

This proves that

$$\overline{\lim_{n=\infty}} c_{\lambda_n}^{1/\lambda_n} \geq 1.$$

That this upper limit can not be greater than unity is a consequence of the fact that c_{λ_n} is the real part of the general coefficient of (10), a series which clearly has unit radius of convergence:

$$\overline{\lim_{n=\infty}} \mid -i\, a_{\lambda_n} e^{2\pi i\psi_n} \mid^{1/\lambda_n} = \overline{\lim_{n=\infty}} \mid a_{\lambda_n} \mid^{1/\lambda_n} = 1.$$

By the Lemma the point $x = 1$ is a singular point of (10). That is, the points

$$x = e^{2\pi i p\phi}, \qquad (p = 1, 2, 3, \cdots),$$

are singular points of (8). Since this set of points is everywhere dense on the circle $\mid x \mid = 1$, it follows that the series (8) admits this circle as a singular curve. A similar proof applies if

$$\overline{\lim_{n=\infty}} a_{\lambda_n}''^{1/\lambda_n} = 1.$$

4. *Applications.* By use of the theorem of section 3 we are now able to prove the following result: *

If $\{\lambda_n\}$ is a sequence of positive integers such that the series

$$\sum_{n=0}^{\infty} 1/\lambda_n$$

converges, then the function defined by

$$\sum_{n=0}^{\infty} a_{\lambda_n} x^{\lambda_n}, \qquad \overline{\lim_{n=\infty}} \mid a_{\lambda_n} \mid^{1/\lambda_n} = 1 \tag{11}$$

has the circle $\mid x \mid = 1$ is a singular curve.

We begin by choosing a subset $\{\mu_n\}$ of the set $\{\lambda_n\}$ for which the unique limit (in contradistinction with the upper limit of the hypothesis), $\lim_{n=\infty} \mid a_{\mu_n} \mid^{1/\mu_n}$, is equal to unity, and for which all the a_{μ_n} lie in a single quadrant. Let ϕ be any irrational number. The set of points $e^{2\pi i\mu_n\phi}$ on the unit circle must contain an infinite number of distinct points, so that there is at least one limit

* Compare Hadamard and Mandelbrojt, *loc. cit.*, p. 71.

point $e^{2\pi i\psi}$. Consequently it is possible to find a subset $\{\nu_n\}$ of the set $\{\mu_n\}$ such that

$$\lim_{n=\infty} e^{2\pi i\nu_n\phi} = e^{2\pi i\psi},$$

$$\lim_{n=\infty} |\, a_{\nu_n} \,|^{1/\nu_n} = 1. \tag{12}$$

Let the set $\{\nu_n'\}$ consist of those values of λ_n not equal to ν_n. If the set $\{\nu_n'\}$ contains only a finite number of elements, the series

$$\sum_{n=0}^{\infty} a_{\nu_n} x^{\nu_n} \tag{13}$$

differs from (11) by the polynomial

$$\sum a_{\nu_n'} x^{\nu_n'}.$$

But (13) satisfies all the conditions of the theorem of section 3, and consequently admits the circle $|\, x \,| = 1$ as a cut. And (13) must have the same singularities as (11), so that the theorem is proved.

On the other hand, if the set $\{\nu_n\}$ contains an infinite number of elements, we set up the entire function

$$g(z) = \prod_{n=0}^{\infty} (1 - z/\nu_n')^2. \tag{14}$$

The series

$$\sum_{n=0}^{\infty} 1/\nu_n'$$

converges, so that the infinite product (14) also converges. Moreover, if ϵ is an arbitrarily small positive constant, $g(z)$ satisfies the inequality

$$|\, g(z) \,| < e^{\epsilon|z|}$$

for all $|\, z \,|$ sufficiently large.* On this account we may apply a theorem of

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* This is a result of the general theory of entire functions. For this special case it may be proved in the following manner: There exists a positive number A such that $|\, 1 - u \,| \leqq e^{A|u|}$ for all complex numbers u. To every positive ϵ there corresponds an integer m such that

$$\sum_{n=m}^{\infty} 1/\nu_n' < \epsilon/4A.$$

Consequently,

$$|\, \prod_{n=m}^{\infty} (1 - z/\nu_n')^2 \,| < e^{\epsilon|z|/2}.$$

But for $|\, z \,|$ sufficiently large

$$|\, \prod_{n=0}^{m-1} (1 - z/\nu_n')^2 \,| < e^{\epsilon|z|/2}.$$

since the modulus of $(1 - z/\nu_n')^2$ increases less rapidly than $e^{\epsilon z/4m}$ as $|\, z \,|$ becomes infinite. The result is thus established.

G. Faber,* and be assured that the series

$$\sum_{n=0}^{\infty} g(\lambda_n)x^{\lambda_n} = \sum_{n=0}^{\infty} g(\nu_n)x^{\nu_n} \tag{15}$$

is an element of a function which has no singularities in the extended plane except the point $x = 1$.

But

$$\sum_{n=0}^{\infty} g(\lambda_n)a_{\lambda_n}x^{\lambda_n} = \sum_{n=0}^{\infty} g(\nu_n)a_{\nu_n}x^{\nu_n}, \tag{16}$$

and, since $g(\nu_n)$ is real and positive, the coefficients all lie in a single quadrant. Moreover, it can be shown that the radius of convergence of (16) is unity. Indeed,

$$\overline{\lim_{n=\infty}}\, g(\nu_n)^{1/\nu_n} = 1$$

since the radius of convergence of (15) is unity. By virtue of (12) we have

$$\overline{\lim_{n=\infty}}\, |\, a_{\nu_n}g(\nu_n)\,|^{1/\nu_n} = \overline{\lim_{n=\infty}}\, |a_{\nu_n}|^{1/\nu_n} = 1.$$

Consequently, the theorem of section 3 is applicable to (16), and implies that the circle $|\,x\,| = 1$ is a cut for this series.

Finally, by an application of Hadamard's † theorem on the multiplication of singularities we see that (11) has all the singularities of (16), and the theorem is completely established.

As an immediate corollary to this theorem we prove the following result, mentioned in the introduction:

If $\{\lambda_n\}$ is a sequence of positive integers such that

$$(\lambda_{n+1} - \lambda_n)/\lambda_n{}^{\sigma}, \qquad 0 < \sigma \leq 1,$$

is greater than some positive constant for all values of n, then the series (11) *admits its circle of convergence as a cut.*

For, the inequality

$$(\lambda_{n+1} - \lambda_n)/\lambda_n{}^{\sigma} > M > 0$$

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* G. Faber, "Über die Fortsetzbarkeit gewisser Taylorscher Reihen," *Mathematische Annalen*, Vol. 57 (1903), p. 369.

† J. Hadamard, "Théorème sur les séries entières," *Acta Mathematica*, Vol. 22 (1898), p. 55.

evidently implies the following inequalities:

$$\lambda_{n+1} > \lambda_n + n^\sigma M,$$
$$\lambda_{n+1} > M\,[n^\sigma + (n-1)^\sigma + \cdots + 2^\sigma + 1],$$
$$\lambda_{n+1} > (M/n^{1-\sigma})[n + (n-1) + \cdots + 2 + 1],$$
$$\lambda_{n+1} > Mn(n+1)/2n^{1-\sigma}.$$

Consequently, we have

$$\sum_{n=0}^{\infty} 1/\lambda_{n+1} < (2/M) \sum_{n=0}^{\infty} 1/n^{1-\sigma},$$

and we are in the situation imposed by the hypotheses of the theorem of this section.

In conclusion, we note that the set $\lambda_n = n!$ provides an example of a set of integers satisfying the conditions of the theorem of section 3. For, take $\phi = e$. Then

$$n!\,e = I_n + R_n,$$

$$R_n = 1/(n+1) + 1/(n+1)(n+2) \cdots < 2/(n+1),$$

where I_n is an integer. Hence

$$e^{2\pi i n!e} = e^{2\pi i R_n},$$

$$\lim_{n=\infty} e^{2\pi i n!e} = 1.$$

THE RICE INSTITUTE,
June, 1927.

A Mathematical Theory of Depreciation and Replacement.

By C. F. Roos.*

1. *Introduction.* In a previous paper I have formulated a problem of depreciation in economics as a Lagrange problem in the calculus of variations and shown that the resulting problem is sufficiently general to include as special cases all of the existing depreciation theories, i. e. such theories as the straight line, the compound interest, the sinking fund, the unit cost plus and that of Hotelling.†

Inasmuch as a machine is seldom used until its useful life has ended, a more important problem is that of determining the time at which a machine in operation should be replaced by another machine having a different operating expense so that a maximum profit is obtained for some period of time extending from an initial time t_1 through a replacement time ω to some final time t_2 greater than ω. When formulated as a problem in the calculus of variations this replacement problem is a type of Lagrange problem with variable end-points and discontinuous integrand.‡

In this paper it is intended to formulate a replacement problem as a general problem in the calculus of variations and to solve in some detail a special example believed to be typical of an actual economic situation. In order to save space I will not attempt to give the whole economic setting of the problem since this can be obtained from other papers.§

2. *The general problem.* The value of a machine to its operator at a time t_1 is the sum of the anticipated rentals which it will yield from the time

* National Research Fellow in Mathematics.

† Roos, "The Problem of Depreciation in the Calculus of Variations," to be offered to the *Bulletin of the American Mathematical Society*. See, also, H. Hotelling, "A General Mathematical Theory of Depreciation," *Journal of American Statistical Association*, Sept. 1925.

‡ Bliss, *Lectures on the Calculus of Variations*, University of Chicago, Summer 1925, mimeographed by O. E. Brown, Northwestern University, Evanston, Ill. See, also, E. H. Clarke, *On the Minimum of the Sum of a Definite Integral and a Function of a Point*, Doctoral Dissertation, University of Chicago.

§ For a brief setting of the problem see footnote (†) above. For a more detailed economic account see Roos, "A Dynamical Theory of Economics," *Journal of Political Economy*, Vol. 35 (Oct. 1927).

t_1 to a time ω at which it is to be salvaged each multiplied by a discount factor to allow for interest plus the salvage value also discounted. If we let $\psi_1(u, u', p, p', t)$ be the cost of operating a machine producing $u(t)$ units of output per unit time which sell at a price $p(t)$; $D_1(u, u', p, p', t)$ be the rate of depreciation in market value of the machine, and K_1 be the cost of the machine at t_1, then, as I have shown,* the value $V_1(t_1)$ of this machine at $t = t_1$ will be

$$(1) \quad V_1(t_1) = \int_{t_1}^{\omega} [pu - \psi_1(u, u', p, p', t) - D_1(u, u', p, p', t)]\, E(t_1, t)\, dt + K_1 E(t_1, \omega).$$

Here and following $E(a, b)$ stands for $\exp\left[-\int_a^b \delta(\nu)\, d\nu\right]$, where $\delta(\nu)$ is the rate of increase of an invested sum s divided by s. The value $V_2(t_1)$ at the time t_1 of a machine which will replace this machine at the time ω will be

$$(2) \quad V_2(t_1) = \int_{t_2}^{\omega} [pu - \psi_2(u, u', p, p', t) - D_2(u, u', p, p', t)]\, E(t_1, t)\, dt + K_2 E(t_1, t_2)$$

where ψ_2, D_2 and K_2 are defined in a way analogous to ψ_1, D_1 and K_1, respectively. For convenience in notation let us write $Q_i = \psi_i + D_i$, $(i = 1, 2)$.

As a likely hypothesis we may suppose that the operator will endeavour to maximize the expression

$$(3) \quad [V_1 + V_2 - K_1 E(t_1, \omega) - K_2 E(t_1, t_2)] = \int_{t_1}^{\omega} [pu - Q_1] E(t_1, t)\, dt + \int_{\omega}^{t_2} [pu - Q_2]\, E(t_1, t)\, dt$$

which is the sum of the values V_1 and V_2 at t_1 minus the respective cost values discounted to take care of interest. In reality K_1 is the sum necessary to replace the first machine by an identical machine at the time ω, and hence

$$(4) \quad K_1(\omega) = K_1(t_2)\, E(\omega, t_2).$$

The preceding hypothesis is, therefore, justified by the fact that if (4) holds, and, furthermore, if $K_1(t_2) \equiv K_2(t_2)$ and $Q_1(u, u', p, p', t) \equiv Q_2(u, u', p, p', t)$, the expression (3) reduces to the corresponding expression for the case of a single machine. In fact, for this special case the replacement problem is actually the problem of depreciation.

We may suppose the rate of production $u(t)$ and the selling price, $p(t)$, of the commodity produced to be related by a differential equation of demand,†

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* Roos, "The Problem of Depreciation in the Calculus of Variations," *loc. cit.*

† Roos, "Dynamical Economics," *Proceedings of the National Academy of Sciences*, Vol. 13, No. 3, pp. 145-150 (March 1927.)

$$\phi(u, u', p, p', t) = 0, \qquad (t_1 \leq t \leq t_2), \tag{5}$$

and to satisfy end conditions,

$$\theta_\mu(t_1, u(t_1), p(t_1), t_2, u(t_2), p(t_2)) = 0, \quad (\mu = 1, \cdots, n \leq 6). \tag{6}$$

We may then state the replacement problem as that of: *Finding among the arcs* $u(t)$, $p(t)$, *satisfying an equation of demand* (5) *and end conditions* (6), *a set which maximizes an integral,* $J = I_1 + I_2$, *where*

$$I_1 = \int_{t_1}^{\omega} [pu - Q_1(u, u', p, p', t)]\, E(t_1, t)\, dt$$

and

$$I_2 = \int_{\omega}^{t_2} [pu - Q_2(u, u', p, p', t)]\, E(t_1, t)\, dt.$$

This problem differs from the depreciation problem in that the integrand of the integral to be maximized is discontinuous at some time ω when the change in machines is to be made.

3. *Conditions for a solution.* Let P_1 and P_2 be the end points of the curve which maximizes J, and let P_ω be the corner point of this maximizing curve. Then it is evident that along the segment P_1P_ω the Euler-Lagrange equations and other conditions of the calculus of variations for the Lagrange problem with variable end-points must hold, for, otherwise, it would be possible to introduce a variation of P_1P_ω which would give a larger value of the integral I_1 taken along it and consequently, a larger value to J, thus contradicting the assumption that P_1P_ω and $P_\omega P_2$ together give J a maximum value. A similar argument holds for the arc $P_\omega P_2$.

In writing these conditions it will be convenient to introduce the notations $y_1(t) = u(t)$, $y_2(t) = p(t)$ and

$$F_\sigma(y_1, y_1', y_2, y_2', t, t_1, \lambda_\sigma)$$

$$= [y_1 y_2 - Q_\sigma(y_1, y_1', y_2, y_2', t)]\, E(t_1, t) + \lambda_\sigma(t)\phi(y_1, y_1', y_2, y_2', t);$$

$$H(t_\sigma) = (\sigma - 2)F_1(t_1) + (\sigma - 1)F_2(t_2) + \int_{t_1}^{\omega} [F_{1t_\sigma} + F_{1y_k(t_\sigma)} y_k'(t_\sigma)]\, dt + \int_{\omega}^{t_2} [F_{2t_\sigma} + F_{2y_k(t_\sigma)} y_k'(t_\sigma)]\, dt;$$

$$H_{y_i}'(t_\sigma) = (\sigma - 2)F_{1y_i}'(t_1) + (\sigma - 1)F_{2y_i}'(t_2) + \int_{t_1}^{\omega} F_{1y_i(t_\sigma)}\, dt + \int_{\omega}^{t_2} F_{2y_i(t_\sigma)}\, dt, \qquad (\sigma = 1, 2;\ i = 1, 2);$$

where k is used as an umbral index with range 1, 2 and subscripts y_i, y_i', $(i = 1, 2)$, and t_σ, $(\sigma = 1, 2)$, denote differentiation.

Then the maximizing arc must satisfy the following conditions:

(a) the Euler-Lagrange equations

$$F_{1y_i}' = \int_{t_1}^{t} F_{1y_i} dt + C_{1i}, \qquad (t_1 \leq t \leq \omega), \quad \text{and}$$

$$F_{2y_i}' = \int_{t_2}^{t} F_{2y_i} dt + C_{2i}, \qquad (\omega \leq t \leq t_2);$$

(b) the transversality conditions which require that every determinant of order $n + 1$ of the matrix:

$$\left\| \begin{matrix} H(t_1) - \underset{\theta_{\mu t_1}}{y_k'(t_1) H_{y_k}'(t_1)} & \underset{\theta_{\mu y_i(t_1)}}{H_{y_i}'(t_1)} & H(t_2) - \underset{\theta_{\mu t_2}}{y_k'(t_2) H_{y_k}'(t_2)} & \underset{\theta_{\mu y_i(t_2)}}{H_{y_i}'(t_2)} \end{matrix} \right\|$$

vanish;

(c) certain second order conditions of the calculus of variations which we will not give here;

(d) at the corner point P_ω the following corner conditions

$$F_1(1) - F_2(2) - \dot{y}_k'(\omega) F_{1\dot{y}_k}'(1) + \bar{y}_k'(\omega) F_{2\bar{y}_k}'(2) + \int_{t_1}^{\omega} F_{1\omega} dt + \int_{\omega}^{t_2} F_{2\omega} dt = 0, \; k \text{ umbral as before,}$$

$$F_{1\dot{y}_\sigma}'(1) - F_{2\bar{y}_\sigma}'(2) + \int_{t_1}^{\omega} F_{1\dot{y}_\sigma(\omega)} dt + \int_{\omega}^{t_2} F_{2\bar{y}_\sigma(\omega)} dt = 0, \quad (\sigma = 1, 2),$$

where $F_1(1)$ means $F_1(y_1, y_1', y_2, y_2', t, t_1, \lambda_1)$ with the arguments y_1, y_1', y_2, y_2', λ_1 replaced by the extremal set $\dot{y}_1$, $\dot{y}_1'$, $\dot{y}_2$, $\dot{y}_2'$, λ_1 for the arc P_1P_ω; $F_2(2)$ means $F_2(y_1, y_1', y_2, y_2', t, t_1, \lambda_2)$ with the arguments y_1, y_1', y_2, y_2', λ_2 replaced by the extremal set $\bar{y}_1$, $\bar{y}_1'$, $\bar{y}_2$, $\bar{y}_2'$, λ_2 for the arc $P_\omega P_2$, and $F_{1y_\sigma}'(1)$ etc. are defined in an analogous manner.*

The corner conditions (d) are, of course, only first necessary conditions. The condition that the second variation at the corner point be not positive is a further necessary condition for a maximum, but we will not attempt to derive this second order condition.†

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* Roos, "A General Problem of Minimizing an Integral with Discontinuous Integrand," *Transactions of the American Mathematical Society*, not yet published. See, also, E. H. Clarke, *On the Minimum of the Sum of a Definite Integral and a Function of a Point, loc. cit.*

† Dresden, *Transactions of the American Matehmatical Society*, Vol. IX, p. 472 (1908), gives the second order corner conditions for the simplest problems of the calculus of variations.

The time $t = \omega$ at which the replacement should occur and the rate of production and price at that time can be determined from the three corner conditions (d). The end times t_1 and t_2 and the end values $y_1(t_1)$, $y_2(t_1)$, $y_1(t_2)$ and $y_2(t_2)$ can be determined from the n end conditions $\theta_\mu = 0$, $(\mu = 1, \cdots, n \leq 6)$ and the $6 - n$ transversality conditions (b).

4. *A replacement example.* Although the theory of the Lagrange problem in the calculus of variations is fairly well developed, there are known comparatively few problems for which the mathematical analysis is sufficiently simple to permit the actual determination of the maximizing or minimizing curves. For this reason it will be all the more interesting and instructive to consider a replacement example which seems to be sufficiently general to typify many economic situations and yet which is sufficiently simple to admit a ready solution.

In order to obtain an expression for the operating expense of a machine or other property we may as well follow the example set by Walras and define certain quantities called coefficients of production.* These quantities are defined as the quantities of the services of the factors of production, i. e. services of land, services of persons and services of capital that enter into the manufacture of a unit of a given commodity. As I have already pointed out, there seems to be justification for writing the coefficients of production for a commodity C as functions of the rate of production of C, the selling price of C and the first derivatives of these quantities.†

If there is one producer manufacturing an amount $u(t)$ of C in unit time and if, furthermore, the selling price of one unit of C is $p(t)$, the coefficients of production are functions, $f_\alpha(u, u', p, p', t)$, $(\alpha = 1, \cdots, m)$, where m is the number of services required to manufacture one unit of C and primes denote derivatives with respect to time. If we denote the prices of the m services required to produce one unit of C by $p_\alpha(t)$, then by the definition of the coefficients of production the cost of producing $u(t)$ units of C for the first or operating machine is

$$\psi_1(u, u', p, p', t) = \sum_{\alpha=1}^{m} u(t) f_{\alpha 1}(u, u', p, p', t) p_\alpha(t).$$

We may suppose the prices of the services $p_\alpha(t)$ to be known functions of the time.

As an approximation we may assume that the depreciation Δ_1 of the operating machine can be represented by $\Delta_1 = \xi_{11} u(t) + \xi_{12} + \xi_{13} u'(t) u'(t)$

* V. Pareto, *Manuel d'Economie Politique*, pp. 607.

† Roos, "A Dynamical Theory of Economics," *loc. cit.*

$+ \xi_{14}p'(t)p'(t) + \xi_{15}u'(t) + \xi_{16}p'(t)$, where $\xi_{11}, \cdots, \xi_{16}$ are known functions of the time or constants. We may further suppose that the coefficients of production f_{a1} depend only upon the rate of production, but, since there is an overhead expense when no units are produced, we would not be justified in assuming the f_{a1} to be linear functions of the rate of production. We may, however, suppose them to have the form $f_{a1} = a_{a1}u(t) + b_{a1} + c_{a1}/u(t)$, where a_{a1}, b_{a1} and c_{a1} are known functions of the time or constants.

We may, therefore, write the function Q_1, which is the cost of production including depreciation in market value of the machine, as

$$Q_1 = f_{a1}p_{a1}(t) + \Delta_1 = A_1u^2 + B_1u + C_1 + D_1u'u' + E_1p'p' + F_1u' + G_1p', \tag{7}$$

where

$$A_1 = a_{a1}p_a, \quad B_1 = \xi_{12} + b_{a1}p_a, \quad C_1 = \xi_{12} + c_{a1}p_a, \quad D_1 = \xi_{13}, \quad E_1 = \xi_{14}, \quad H_1 = \xi_{15}, \quad G_1 = \xi_{16}$$

and α is an umbral index with range $1, \cdots, m$ according to the convention that whenever a literal suffix appears twice in a term that term is to be summed for all values of the suffix.* The quantities $A_1, \cdots, G_1$ are, therefore, known functions of the time or constants.

By a similar argument we may write for the replacement machine

$$Q_2 = f_{a2}p_{a2}(t) + \Delta_2 = A_2u^2 + B_2u + C_2 + D_2u'u' + E_2p'p' + F_2u' + G_2p', \tag{8}$$

where $A_2, \cdots, G_2$ are known functions of the time, analogous, respectively, to $A_1, \cdots, G_1$ defining Q_1.

If we suppose the rate of demand to be a linear function of the price and the rate of change of price, and, furthermore, that as many units are sold as are produced, the demand equation of (5) becomes

$$u - a(t)p(t) - h(t)p'(t) - b(t) = 0, \tag{9}$$

where a, b and h are known functions of the time or constants.
When the price p and the rate of production u are known at $t_1 = \tau_1$, the equations $\theta_\mu = 0$, $\mu = 1, 2, 3$, of (6) become

$$\begin{aligned} \theta_1 &= t_1 - \tau_1 = 0 \\ \theta_2 &= u(\tau_1) - u_1 = 0 \\ \theta_3 &= p(\tau_1) - p_1 = 0. \end{aligned} \tag{10}$$

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* A. S. Eddington, "The Mathematical Theory of Relativity," p. 50, 22.1.

The operator of the machine will desire to choose the rate of production $u(t)$ and the price $p(t)$, satisfying a differential equation of demand (9) and having end values satisfying the end conditions (10), so that they maximize the integral

$$(11)\quad J = \int_{t_1}^{\omega} [pu - A_1u^2 - B_1u - C_1 - D_1u'u' - E_1p'p' - H_1u' - G_1p']E(t_1, t)\,dt$$
$$+ \int_{\omega}^{t_2} [pu - A_2u^2 - B_2u - C_2 - D_2u'u' - E_2p'p' - H_2u' - G_2p']E(t_1, t)\,dt.$$

We first consider the part of the arc which lies between $t = \tau_1$ and $t = \omega$, i. e. what would correspond to P_1P_ω of section (3). Since the function F_1 of section (3) is $F_1 = [pu - A_1u^2 - B_1u - C_1 - D_1u'u' - E_1p'p' - H_1u' - G_1p' + \lambda_1(t)(u - ap - hp' - b)]E(t_1, t)$, we have *

$$\partial F_1/\partial p = [u - a\lambda_1]E(t_1, t); \qquad \partial F_1/\partial p' = [2E_1p' - G_1 - h\lambda_1]E(t_1, t);$$
$$\partial F_1/\partial u = [p - 2A_1u - B_1 + \lambda_1]E(t_1, t); \qquad \partial F_1/\partial u' = [2D_1u' - H_1]E(t_1, t).$$

The Euler-Lagrange equations (a) in the form $\partial F_1/\partial p = (d/dx)\partial F_1/\partial p'$, etc., obtained by differentiating (a) with respect to t, become on cancelling the factor $E(t_1, t)$

$$u - a\lambda_1 = [2E_1p' - G_1 - h\lambda_1]\delta + 2E_1'p' + 2E_1p'' - G_1' - h'\lambda_1 - h\lambda_1'$$
$$p - 2A_1u - B_1 + \lambda_1 = [2D_1'u' - H_1]\delta + 2D_1'u' + 2D_1u'' - H_1'.$$

It is possible to carry the problem through as it is, but since we do not want to lose the reader in a mass of detail, we will assume $A_1, \cdots, G_1, a, b, h$ and δ to be constant. We may then write the last equations as

$$(11A)\quad u - a\lambda_1 = (2E_1p' - G_1 - h\lambda_1)\delta + 2E_1p'' - h\lambda_1'$$

$$(11B)\quad p - 2A_1u - B_1 + \lambda_1 = (2D_1u' - H_1)\delta + 2D_1u''.$$

If we solve the second of these equations for λ_1 and substitute the resulting value of λ_1 and $\lambda_1' = d\lambda_1/dt$ in the first, we obtain on collecting terms

$$(12)\quad (a - h\delta)(B_1 - H_1\delta) + G_1\delta + [1 - (a - h\delta)2A_1]u$$
$$+ [2A_1h - (a - h\delta)2D_1\delta]u' + [2D_1h\delta - (a - h\delta)D_1]u''$$
$$+ D_1hu''' + (a - h\delta)p - (2\delta E_1 - h)p' - 2E_1p'' = 0.$$

The demand equation, $u = ap + b + hp'$, gives us by differentiation $u' = ap' + hp''$, $u'' = ap'' + hp'''$ and $u''' = ap''' + hd^4p/dt^4$. When we substitute these values of $u, \cdots, u'''$ in (12), we obtain

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* We have chosen λ_1 as $\lambda_1E(t_1, t)$. Since t_1 is fixed this choice does not introduce any new complications and allows us to cancel the factor $E(t_1, t)$.

$$(13)\quad L_{14}d^4p/dt^4 + L_{13}d^3p/dt^3 + L_{12}d^2p/dt^2 + L_{11}dp/dt + L_{10}p + p_{10} = 0,$$

where

$$\begin{aligned}
L_{14} &= hD_1,\\
L_{13} &= 3D_1h\delta - aD_1 + hD_1,\\
L_{12} &= 2A_1h^2 - 2ahD_1 + 2D_1h^2\delta^2 + 3ah\delta D_1 - a^2D_1 - 2E_1,\\
L_{11} &= (2A_1h^2 - 2a^2D_1 + 2ah\delta D_1 - 2E_1)\delta,\\
L_{10} &= 2a(1 - A_1a) - h\delta(1 - 2aA_1),\\
p_{10} &= [1 - (a - h\delta)2A_1]b + (a - h\delta)(B_1 - H_1\delta) + G_1\delta.
\end{aligned}$$

Since (13) is a fourth order differential equation with constant coefficients, its general solution is

$$(14)\qquad p_1 = p_{10} + K_{11}\,e^{m_{11}t} + K_{12}\,e^{m_{12}t} + K_{13}\,e^{m_{13}t} + K_{14}\,e^{m_{14}t},$$

where $m_{11}, \cdots, m_{14}$ are the four roots of the quartic $L_{14}m^4 + L_{13}m^3 + L_{12}m^2 + L_{11}m + L_{10} = 0$ and $K_{11}, \cdots, K_{14}$ are arbitrary constants.

Similarly, if we assume that $A_2, \cdots, G_2$ and δ, for the replacement machine, are constant for the period $\omega \leq t \leq t_2$, we can obtain

$$(15)\qquad p_2 = p_{20} + K_{21}\,e^{m_{21}t} + K_{22}\,e^{m_{22}t} + K_{23}\,e^{m_{23}t} + K_{24}\,e^{m_{24}t},$$

with $p_{20}, \cdots, m_{24}$ defined by replacing the set $A_1, \cdots, G_1$ in $p_{10}, \cdots, m_{14}$ for the operating machine by the corresponding set $A_2, \cdots, G_2$ for the replacement machine.

If we substitute the value of p_1 given by (14) in the equation of demand (9), we obtain on collecting terms

$$(16)\quad u_1 = ap_{10} + b + K_{11}(a + hm_{11})e^{m_{11}t} + \cdots + K_{14}(a + hm_{14})e^{m_{14}t}.$$

By a similar procedure with p_2 we obtain

$$(17)\quad u_2 = ap_{20} + b + K_{21}(a + hm_{21})e^{m_{21}t} + \cdots + K_{24}(a + hm_{24})e^{m_{24}t}.$$

In order to complete the solution of the problem we must determine the constants $K_{11}, \cdots, K_{14}$, $K_{21}, \cdots, K_{24}$, the time t_1, the time ω at which the replacement should occur and the time t_2 at which the replacement machine should be scrapped. To determine these eleven quantities we must make use of the end conditions (10), the transversality conditions (b) and the corner conditions (d). We proceed to set up the matrix (b) for the three end conditions (10) and the function F defined as $F = F_1$ on $t_1 \leq t \leq \omega$ and $F = F_2$ on $\omega \leq t \leq t_1$. We easily obtain the rectangular matrix with four rows and six columns

$$\begin{Vmatrix} S_1 & \partial F_1(t_1)/\partial p_1' & \partial F_1(t_1)/\partial u_1' & S_2 & \partial F_2(t_2)/\partial p_2' & \partial F_2(t_2)/\partial u_2' \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{Vmatrix}$$

where

$$S_\sigma = (-1)^\sigma [F(t_\sigma) - p_\sigma'(t_\sigma)\ \partial F_\sigma(t_\sigma)/\partial p_\sigma' - u_\sigma'\ (t_\sigma)\partial F_\sigma(t_\sigma)/\partial u_\sigma'] + I_\sigma \delta \qquad (\sigma = 1, 2)$$

and $\partial F_\sigma(t_\sigma)/\partial p_\sigma'$ means $\partial F_\sigma/\partial p'$ with p_σ, p_σ', u_σ, u_σ' at t_σ substituted for p, p', u, u'. The value of λ_σ to be used in $\partial F_\sigma/\partial p_\sigma'$ is the value of λ_σ given by (11B) and an analogous equation for λ_2. In (11B) p, u, u', u'' must be replaced by p_1, u_1, u_1', u_1'', and similarly for λ_2.

In order that every determinant of order four of this matrix vanish, it is necessary and sufficient that

$$S_2 = 0, \quad \partial F_2(t_2)/\partial p_2' = 0, \quad \partial F_2(t_2)/\partial u_2' = 0. \tag{18}$$

We have thus three transversality conditions.

From the corner conditions (d) we obtain the three additional conditions

$$\begin{aligned} &F_1(\omega) - F_2(\omega) - p_1'(\omega)\partial F_1(\omega)/\partial p_1' - u_1'(\omega)\partial F_1(\omega)/\partial u_1' \\ &\qquad + p_2'(\omega)\partial F_2(\omega)/\partial p_2' + u_2'(\omega)\partial F_2(\omega)/\partial u_2' = 0, \\ &\partial F_1(\omega)/\partial u_1' - \partial f_2(\omega)/\partial u_2' \equiv 2D_1 u_1'(\omega) - 2D_2 u_2'(\omega) = 0, \\ &\partial F_1(\omega)/\partial p_1' - \partial F_2(\omega)/\partial p_2' \equiv 2E_1 p_1'(\omega) - 2E_2 p_2'(\omega) \\ &\qquad - [\lambda_1(\omega) - \lambda_2(\omega)] = 0, \end{aligned} \tag{18}$$

where $\partial F_1(\omega)/\partial p_1'$ is defined as $\partial F_1(t_1)/\partial p_1'$ with t_1 replaced by ω.

Since we have required our maximizing arc to be continuous at $t = \omega$, the following equations must be satisfied

$$p_1(\omega) - p_2(\omega) = 0, \quad u_1(\omega) - u_2(\omega) = 0. \tag{19}$$

For the determination of the eleven quantities t_1, ω, t_2, $K_{11}, \cdots, K_{14}$, $K_{21}, \cdots, K_{24}$ we have, therefore, eleven equations consisting of the three end conditions (10), the three transversality conditions (17), the three corner conditions (18) and the two continuity conditions (19). As one may readily verify, the equations of the highest degree in t_1, ω, t_2, $K_{11}, \cdots, K_{24}$ will be the transversality equation $S_2 = 0$, and the first corner condition of (18). These equations will be quadratics in $K_{11}, \cdots, K_{24}$ and will contain t_1, ω and t_2 as exponents of e. There will in general, therefore, be more than one solution for the eleven quantities $t_1, \cdots, K_{24}$. To determine which solution maximizes J we need sufficiency conditions for this problem. In an actual

case we would probably be able to fix t_1 and t_2, and this would simplify matters considerably. The condition $S_2 = 0$ would then drop out, and the only quadratic would be the first corner condition. We could then discard all values of ω which did not lie between t_1 and t_2. The discussion of this phase of the problem is, however, beyond the purpose of this paper.

Let us observe that if we fix all end values and take $\delta \equiv D_1 \equiv \cdots \equiv G_1 \equiv D_2 \equiv \cdots \equiv G_2 \equiv 0$ and $A_1 \equiv A_2$, $B_1 \equiv B_2$ and $C_1 \equiv C_2$, our transversality and corner conditions vanish identically and the differential equation (13) reduces to the differential equation obtained by G. C. Evans in a paper which introduced the new dynamical theory of economics.*

5. *Some further generalizations.* The analogue of the Hotelling hypothesis, that the operator of a machine will do everything in his power to maximize the present value $V_1(t_1)$ of his machine, would require for the replacement problem that the operator endeavor to maximize the sum, $V_1(t_1) + V_2(t_1)$ of section 2. This replacement problem does not seem to be a special case of a very general Mayer problem discussed by Bliss, because of the way in which the corner value ω appears.† It is, however, a special case of the general problem to which the conditions (a), (b), (c) and (d) of section 3 belong, for, we can obtain the Euler-Lagrange, transversality and corner conditions by replacing F_1 and F_2 in (a), (b) and (d) of section 3 by

$$G_1 \equiv F_1 + (K_1 + K_2)E(t_1, t)/\omega - t_1 \quad \text{and} \quad G_2 \equiv F_2, \text{ respectively.}$$

I have already discussed a related depreciation problem for the case of several competing machines all of which continue in operation for the period $t_1 \leq t \leq t_2$, and I do not believe that the introduction of the interest factor would unduly complicate matters.‡ The replacement problem for several competing machines is, however, much more difficult because of the fact that all replacements need not occur at the same time ω, i. e. one machine may be replaced at a time ω_1, another at a time ω_2, and so on.

The replacement problem for the case of n cooperating machines, which we may suppose to be replaced at the n time $\omega_1, \omega_2, \cdots, \omega_n$, ought not, however, to impose any serious mathematical complications. For example, if all

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* G. C. Evans, "The Dynamics of Monopoly," *American Mathematical Monthly*, Vol. 31, No. 2 (1924).

† Bliss, "The Problem of Mayer with Variable End-points," *Transactions of the American Mathematical Society*, Vol. 19 (1918).

‡ Roos, "A Mathematical Theory of Competition," *American Journal of Mathematics*, Vol. 47, No. 3 (July 1925).

of the n cooperating machines produced a like commodity, we would need to maximize an expression of the form

$$J = \int_{t_1}^{\omega_1} f_{11}dt + \int_{\omega_1}^{t_2} f_{12}dt + \int_{t_1}^{\omega_2} f_{21}dt + \int_{\omega_2}^{t_2} f_{22}dt + \cdots + \int_{t_1}^{\omega_n} f_{n1}dt + \int_{\omega_n}^{t_2} f_{n2}dt,$$

where the $f_{i\sigma}$ $(i = 1, \cdots, n;\ \sigma = 1, 2)$, are functions of the rates of production $u_1(t_1), \cdots, u_n(t)$ of the n cooperating machines and of the price $p(t)$, satisfying a single equation of demand $\phi(u_1, u_1', u_2, u_2', \cdots, u_n, u_n', p, p', t) = 0$. The problem is, however, too important to be disposed of in this cursory manner.

It is possible that this cooperation problem might yield the key by which it would be possible to consider a competition problem for a variable number of competitors. Certainly, the cooperation problem for a variable number of competitors can be set up by the methods of the preceding paragraph, for, it is the special case of the general cooperation problem of the preceding paragraph when $f_{i1} \equiv 0$.

There is another important possible direction of extension of this paper. As I have already pointed out in another paper, there are instances when a more general equation of demand

$$\phi(u, u', p, p', t) = \int_{t_1}^{t} \theta(u, u', p, p', t, s)ds$$

should replace the differential equation (5) of demand.* This leads to a problem somewhat similar to the problem of maximizing a functional proposed by Hahn.†

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* Roos, "A Dynamical Theory of Economic Equilibrium," *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 280-285.

† H. Hahn, "Ueber die Lagrange'sche Multiplicatorenmethode," *Sitzungsberichte der Akademie Wien*, Vol. 131 (1922), pp. 531-550. See, also, citation (*) above.